Macroscopic dielectric theory

Macroscopic dielectric theory

Maxwell's equations

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

Macroscopic dielectric theory

In a medium it is convenient to explicitly introduce induced charges and currents

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho$$
$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\left(\vec{J} + \vec{J}_{ext}\right) + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{D} = \vec{E} + 4\pi\vec{P}$$
$$\vec{H} = \vec{B} + 4\pi\vec{M}$$

With no external charges and currents

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Electric polarization

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Electric polarization

Magnetic polarization

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Electric polarization ($\vec{P}$)

Magnetic polarization ($\vec{M}$)

Induced current density ($\vec{J}$)

With no external charges and currents

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{D} = 0$$
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Macroscopic dielectric theory

$\vec{P}$, $\vec{M}$ and $\vec{J}$ represent the response of the medium to the electromagnetic field

If we assume the response is linear, we can write:

$$\vec{P} = \alpha\vec{E}$$
$$\vec{J} = \sigma E$$
$$\vec{M} = \chi\vec{B}$$

where α,σ e χ are the polarizability, conductivity and susceptibility tensors, respectively.

For optical frequencies we can neglect susceptibility (χ=0) so that $\vec{B} = \vec{H}$

Macroscopic dielectric theory

By introducing into the fourth of Maxwell's equations we get:

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \left(\sigma \vec{E} + \alpha \frac{\partial \vec{E}}{\partial t} \right) + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

Conductivity (σ) and polarizability (α) both contain the material response giving rise to two induced currents, a polarization current:

$$\vec{J}_{pol} = \alpha \frac{\partial \vec{E}}{\partial t}$$

and a ohmic current

$$\vec{J}_{ohm} = \sigma \vec{E}$$

Macroscopic dielectric theory

The absorbed power per unit volume is given by:

$$\left(\frac{\partial W}{\partial t}\right)_{abs} = \overline{\vec{J}\cdot\vec{E}} = \sigma\overline{E^2}$$

If we consider solutions of Maxwell's equations in the form $\sin(\omega t)$ we see immediately that J_{ohm} is in phase with E while J_{pol} is out phase by π/2. Polarization current does not give rise to absorption and is responsible of *dispersion* (refraction index). Ohmic current, in turn produce an *absorpion*:

$$\left(\frac{\partial W}{\partial t}\right)_{abs} = \overline{\vec{J}_{ohm}\cdot\vec{E}} = \sigma\overline{E^2}$$

Macroscopic dielectric theory

We introduce the dielectric tensor ε

$$\vec{D} = \epsilon \vec{E}$$

$$\epsilon = 1 + 4\pi\alpha$$

As with simple a.c. circuits, absorption and dispersion can be treated with a single *complex* quantity.

If we consider a field in the form

$$\vec{E} = \vec{E}_0 e^{-i\omega t}$$

we get

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{1}{c}\left[\epsilon\frac{\partial \vec{E}}{\partial t} - 4\pi\vec{E}\right] = \frac{\tilde{\epsilon}}{c}\frac{\partial \vec{E}}{\partial t}$$

in which we have introduced the complex dielectric function:

$$\tilde{\epsilon} = \epsilon_1 + i\epsilon_2 = \epsilon + i\frac{4\pi\sigma}{\omega}$$

Macroscopic dielectric theory

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \left(\sigma \vec{E} + \alpha \frac{\partial \vec{E}}{\partial t} \right) + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\begin{aligned} \vec{E} &= \vec{E}_0 e^{-i\omega t} \\ \frac{\partial \vec{E}}{\partial t} &= -i\omega \vec{E}_0 e^{-i\omega t} = -i\omega \vec{E} \end{aligned}$$

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \frac{4\pi}{c} \sigma \vec{E} + \frac{1}{c} (4\pi\alpha + 1) \frac{\partial \vec{E}}{\partial t} = \\ &= \frac{4\pi}{c} \sigma \vec{E} + \frac{\epsilon}{c} \frac{\partial \vec{E}}{\partial t} \\ &= \left[\frac{4\pi}{c} \sigma + (-i\omega) \frac{\epsilon}{c} \right] \frac{i}{\omega} \frac{\partial \vec{E}}{\partial t} = \\ &= \left[\frac{\epsilon}{c} + i \frac{4\pi\sigma}{c} \right] = \frac{\tilde{\epsilon}}{c} \frac{\partial \vec{E}}{\partial t} \end{aligned}$$

Macroscopic dielectric theory

By introducing the complex dielectric constant into Maxwell's equations we get

$$\nabla^2 \vec{E} - \frac{\tilde{\epsilon}}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

As well known a class of solutions of the wave equations are plane waves:

$$\vec{E} = \vec{E}_0 e^{i(\vec{q} \cdot \vec{r} - \omega t)} + c.c.$$

Macroscopic dielectric theory

whose dispersion law

$$\omega^2 = \frac{c^2}{\tilde{\epsilon}} q^2$$

suggest a complex refraction

$$\tilde{n} = n + i\kappa$$

so that

$$\tilde{n}^2 = \tilde{\epsilon}$$

and therefore

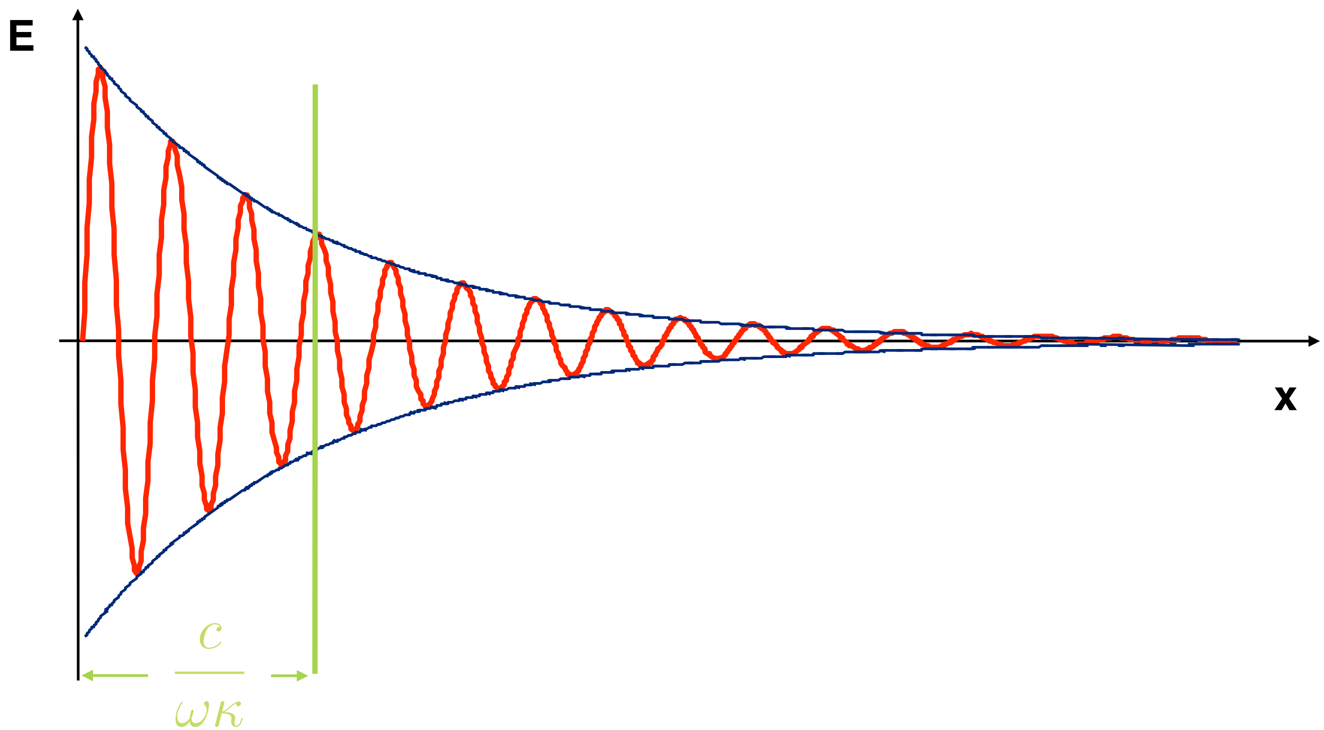
$$\epsilon_1 = n^2 - \kappa^2$$

$$\epsilon_2 = 2n\kappa$$

Macroscopic dielectric theory

A plane wave propagating along the x axis can be written in terms of the complex dielectric index:

$$\vec{E} = \vec{E}_0 e^{i(\vec{q}\cdot\vec{r} - \omega\, t)} + c.c. = \vec{E}_0 e^{-i\omega\left(t - \frac{\tilde{n}x}{c}\right)} + c.c. = \vec{E}_0 e^{-\frac{\omega\kappa x}{c}} e^{-i\omega\left(t - \frac{nx}{c}\right)} + c.c.$$



Phase and group velocity

$$v_p = \frac{\omega}{k} = \frac{\lambda}{T}$$

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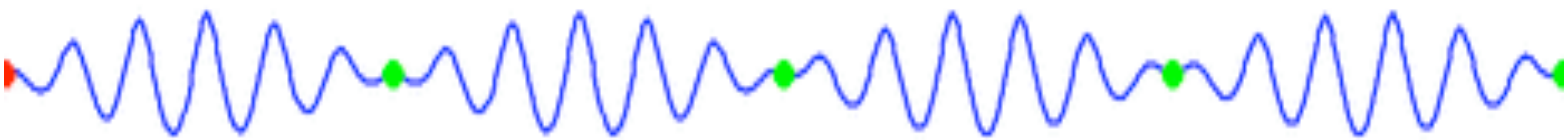


Macroscopic dielectric theory

The absorption coefficient η is defined by the equation:

$$\overline{W} = \overline{W}_0 e^{-\eta x}$$

which describes the attenuation of the average energy flux, which is proportional to the square of the field , so:

$$\eta = \frac{2\omega\kappa}{c} = \frac{\omega\epsilon_2}{nc} = \frac{4\pi\sigma}{nc}$$

Macroscopic dielectric theory

The complex reflection coefficient is given by the generalization of Frenel law:

$$r = \frac{\tilde{n} - 1}{\tilde{n} + 1} = \frac{(n-1) + i\kappa}{(n+1) + i\kappa}$$

So the reflectivity is:

$$R = r^* r = \frac{(n-1)^2 + \kappa^2}{(n+1)^2 + \kappa^2}$$

If $\kappa \gg n$, $R \approx 1$ and thw wave is rapidly attenuated in the medium. This happens, for example in metals at low frequency (high σ)

Macroscopic dielectric theory

The real and imaginary part of the dielectric function and in general of any function describing the linear response to a physical stimulus are **not** independent of each other. This is a consequence of the **causality principle,** which states that the response cannot occur in time preceding the stimulus.

Let us focus on polarization (response) and the electric field (stimulus).
The following dependence will hold:

$$\vec{P}\,(t) = \int_{-\infty}^{t} G\,(t - t')\,\vec{E}\,(t')\,dt'$$

i.e. the polarization at a given time *t* depends on the history of the electric field weighted by the Green function *G.*

Macroscopic dielectric theory

For a sinusoidal field (and a substitution of a variable) we ca write:

$$\vec{P}\left(t\right)=\int_{-\infty}^{t}G\left(t-t'\right)\vec{E}_0e^{-i\omega t'}\left(t'\right)dt'=\vec{E}_0e^{-i\omega t}\int_{0}^{\infty}G\left(\tau\right)e^{i\omega\tau}d\tau$$

so we the complex response function (polarizability in this case) can **always** be expressed as:

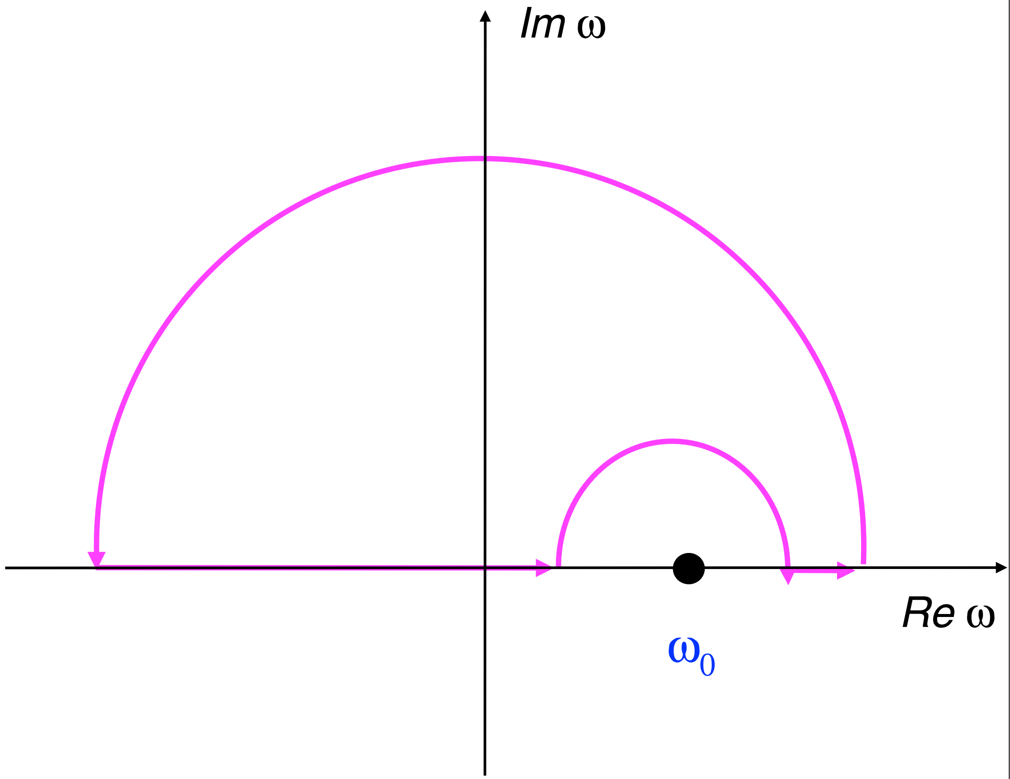
$$\tilde{\alpha}\left(\omega\right)=\int_{0}^{\infty}G\left(\tau\right)e^{i\omega\tau}d\tau$$

Macroscopic dielectric theory

If $\tilde{\alpha}(\omega)$ decreases to zero away from the real axis, the function $\alpha(\omega)/(\omega-\omega_0)$ has a single pole in ω_0.

And its integral along the path in the figure is zero, i.e. by applying Cauchy's theorem:

$$P\int_{-\infty}^{+\infty} \frac{\alpha(\omega)}{\omega-\omega_0} d\omega = i\pi\alpha(\omega_0)$$



where P indicates the principal part of the integral.
For the dielectric function we obtain the dispersion relation:

$$\epsilon(\omega) = 1 + \frac{1}{i\pi} P\int_{-\infty}^{+\infty} \frac{\epsilon(\omega')-1}{\omega'-\omega} d\omega'$$

Macroscopic dielectric theory

From:

$$\tilde{\alpha}(\omega) = \int_0^\infty G(\tau)\, e^{i\omega\tau} d\tau$$

we immediately obtain that

$$\epsilon(-\omega) = \epsilon^*(\omega)$$

we can therefore rewrite the integral and write the Kramers-Kronig dispersion relations between the real and imaginary part of the dielectric function:

$$\epsilon_1(\omega) = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \epsilon_2(\omega')}{\omega'^2 - \omega^2} d\omega'$$

$$\epsilon_2(\omega) = -\frac{2\omega}{\pi} P \int_0^\infty \frac{\epsilon_1(\omega') - 1}{\omega'^2 - \omega^2} d\omega'$$

Dispersion relation for reflectivity

$$r(\omega) = \frac{(n-1)+i\kappa}{(n+1)+i\kappa} = |r(\omega)|\, e^{i\theta}$$

$$\mathsf{R}(\omega) = |r(\omega)|^2 = \left|\frac{(n-1)+i\kappa}{(n+1)+i\kappa}\right|^2$$

$$\ln(r(\omega)) = \ln(|r(\omega)|) + i\theta(\omega) = \frac{1}{2}\ln(\mathsf{R}(\omega)) + i\theta(\omega)$$

$$\theta(\omega) = -\frac{2\omega}{\pi} P\int_0^\infty \frac{\ln(|r(\omega')|)}{\omega'^2-\omega^2}\, d\omega' = -\frac{\omega}{\pi} P\int_0^\infty \frac{\ln(\mathsf{R}(\omega'))}{\omega'^2-\omega^2}\, d\omega'$$

Dispersion relation for reflectivity

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We measure:

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If we consider the natural logarithm of r(ω):

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Macroscopic dielectric theory

Static sum rule.
For ω=0 we have:

$$\epsilon_1(0) = 1 + \frac{2}{\pi} P \int_0^{\infty} \frac{\epsilon_2(\omega')}{\omega'} d\omega'$$

so, if the static dielectric constant is ≠1, the imaginary part wil be ≠0 in some part of the spectrum, i.e the medium must absorb radiation. Because of the $1/\omega$ factor in the integrand, the static dielectric function will be greater if the absorption occurs at low frequency.

Macroscopic dielectric theory

Absorption localized at $\omega = \omega_0$

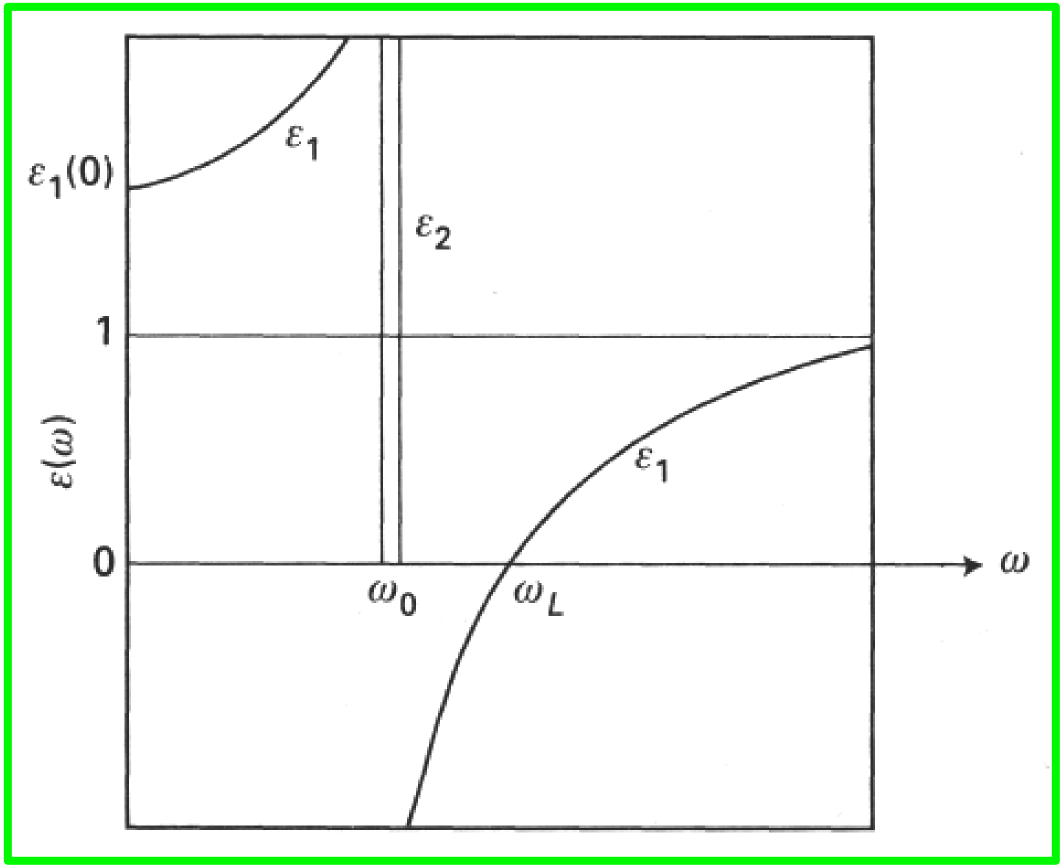
$$\epsilon_2(\omega) = A\delta(\omega - \omega_0)$$

$$\epsilon_1(0) = 1 + \frac{2}{\pi}\frac{A}{\omega_0}$$

therefore:

$$\epsilon_1(\omega) = 1 + \frac{\omega_0^2\left[\epsilon_1(0) - 1\right]}{\omega_0^2 - \omega^2}$$

$$\epsilon_2(\omega) = \frac{\pi}{2}\omega_0\left[\epsilon_1(0) - 1\right]\delta(\omega - \omega_0)$$



Macroscopic dielectric theory

Drude-Lorentz model.

We describe the medium as an ensemble of harmonic oscillators whose resonance frequency and damping are ω_0 and γ, respectively:

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We obtain the dipole moment per unit volume by multiplying y by the charge e and by the oscillator density *N*. The dielectric function is therefore:

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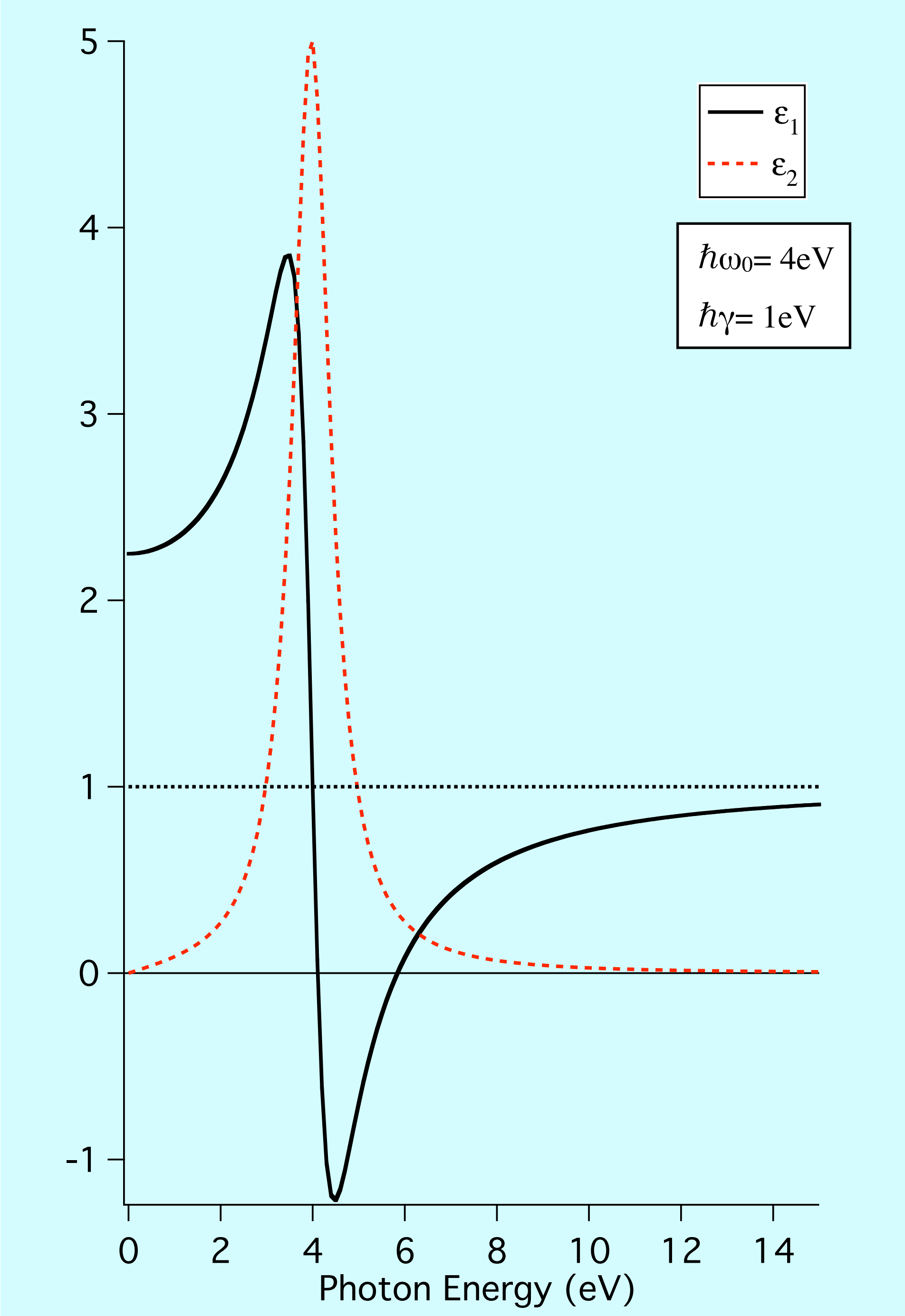
$$\tilde{\epsilon}\left(\omega\right) = 1 + 4\pi\tilde{\alpha} = 1 + 4\pi\frac{P}{E} = 1 + \frac{4\pi e^2 N}{m}\frac{1}{\left(\omega_0^2 - \omega^2\right) - i\gamma\omega}$$

Macroscopic dielectric theory

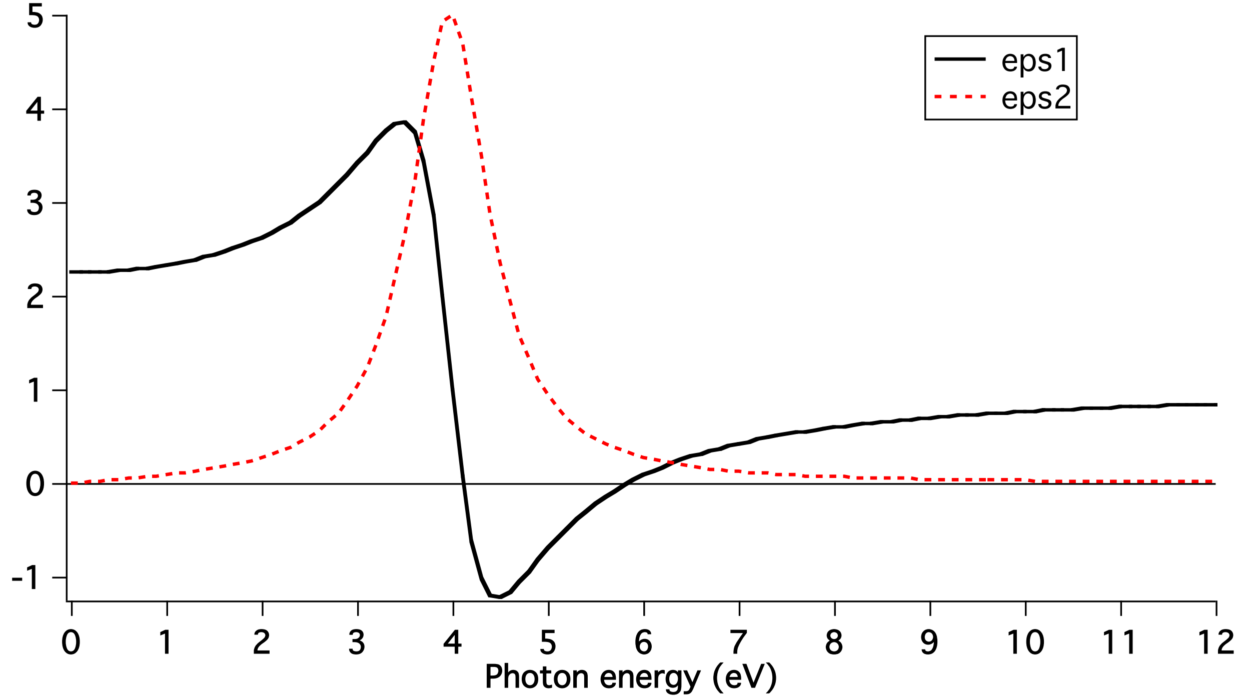
So:

$$\epsilon_1 = 1 + \frac{4\pi e^2 N}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$

$$\epsilon_2 = \frac{4\pi e^2 N}{m} \frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$



Macroscopic dielectric theory

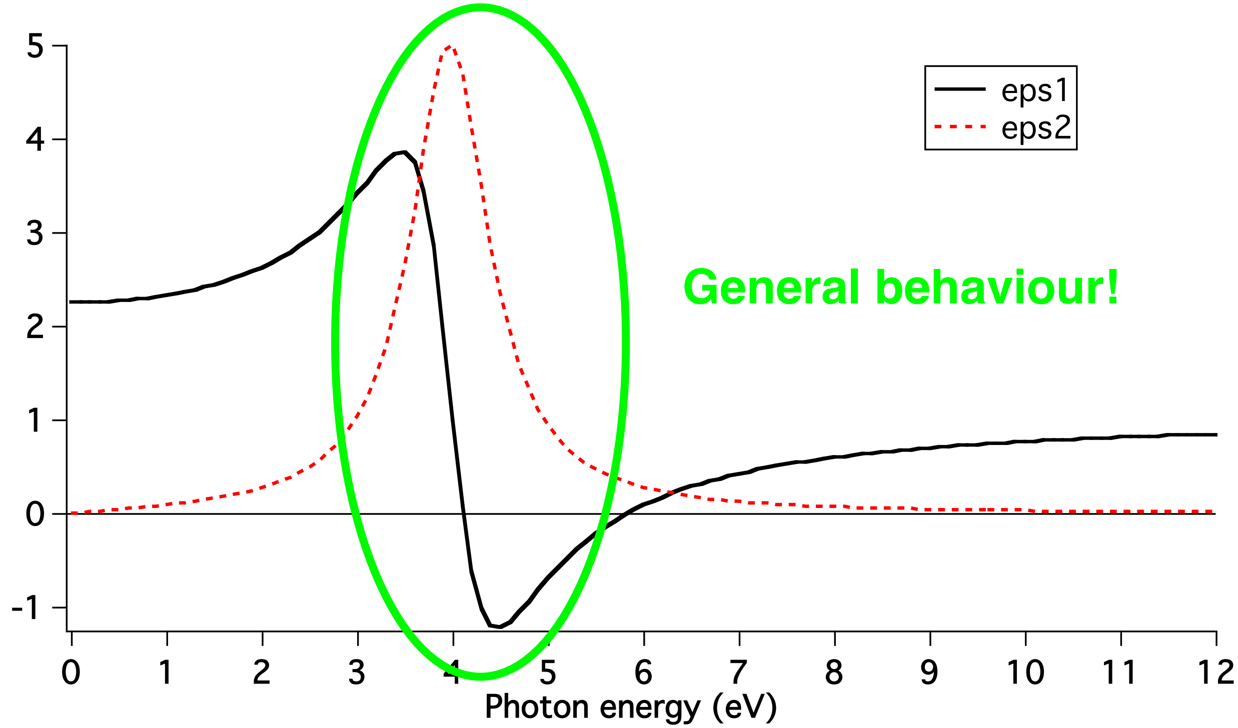


$$\epsilon_2(\omega) = -\frac{2\omega}{\pi} P\int_0^\infty \frac{\epsilon_1(\omega') - 1}{\omega'^2 - \omega^2} d\omega'$$

Integration by parts

$$\epsilon_2(\omega) = -\frac{1}{\pi} P\int_0^\infty \left[\frac{d\epsilon_1(\omega')}{d\omega'}\right] \cdot \ln\left[\frac{\omega' + \omega}{\omega' - \omega}\right] d\omega'$$

Macroscopic dielectric theory



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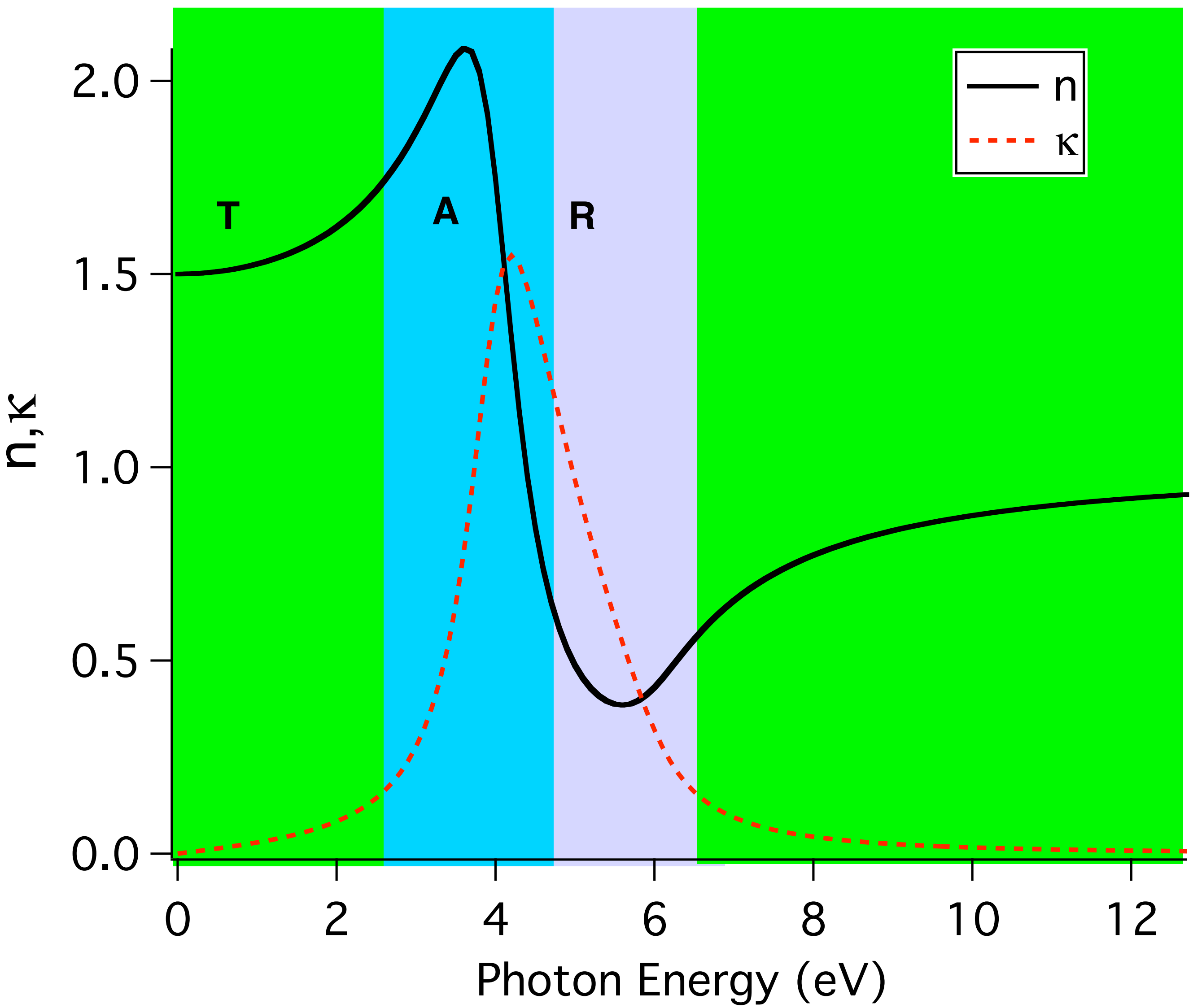
Macroscopic dielectric theory

By considering the complex refraction index, one can identify regions in which transmissivity, absorption or reflectivity are the main effects:



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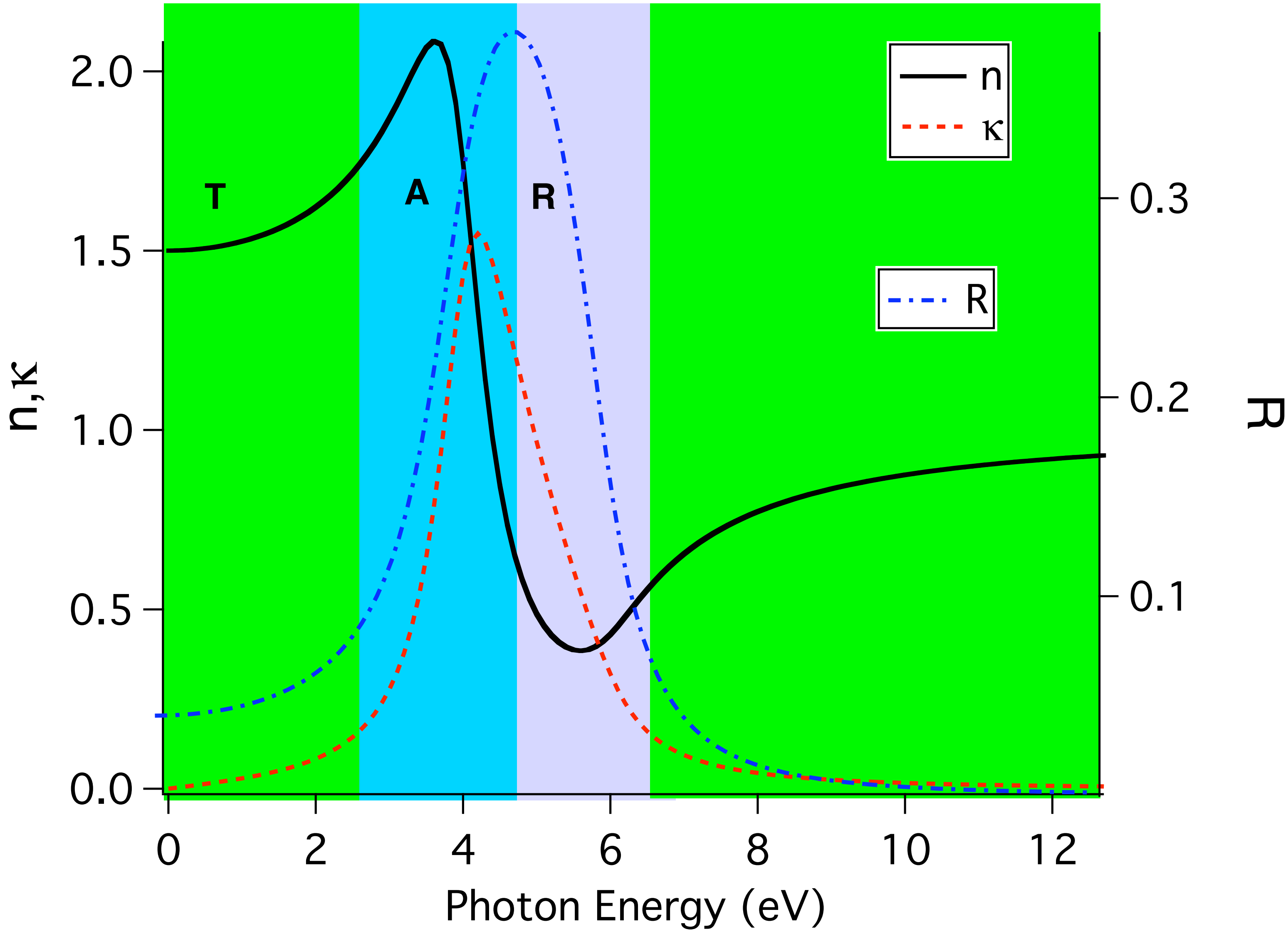


Macroscopic dielectric theory

Optical properties of metals can be obtained from the Lorentz-Drude model by setting ω_0=0 and by defining the *plasma frequency:*

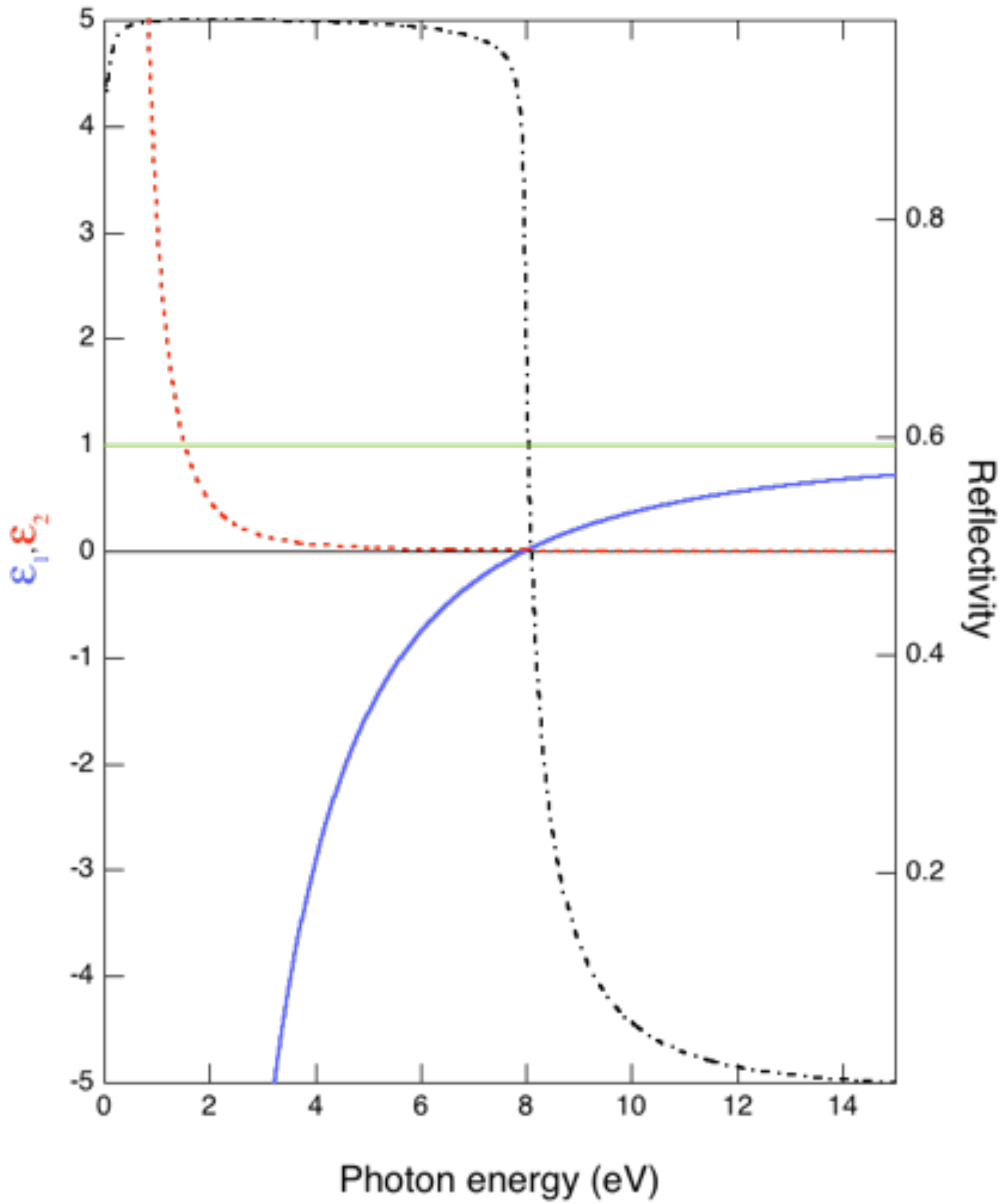
$$\omega_p^2 = \frac{4\pi N e^2}{m}$$

The dielectric functions becomes:

$$\tilde{\epsilon} = \left(1 - \frac{\omega_p^2}{\omega^2 + \gamma^2}\right) + i\frac{\gamma\omega_p^2}{\omega^3 + \gamma^2\omega}$$

Macroscopic dielectric theory

Free electron metal with
$\hbar\omega_p$=8eV and $\hbar\gamma$=0.5 eV



Macroscopic dielectric theory

The conductivity, σ is given by:

$$\sigma = \frac{\omega \epsilon_2}{4\pi}$$

which in the statc limit becomes:

$$\sigma_0 = \lim_{\omega \to 0} \frac{\omega \epsilon_2}{4\pi} = \frac{Ne^2}{m\gamma}$$

which can be compared with the expression derived in the transport theory to find that $\gamma = 1/\tau$

Macroscopic dielectric theory

At very low frequencies ($\omega << 1/\tau$) the reflectivity is:

$$R \cong 1 - 2\sqrt{\frac{\omega}{2\pi\sigma}} = 1 - 2\sqrt{\frac{2\omega}{\omega_p^2\tau}}$$

Macroscopic dielectric theory

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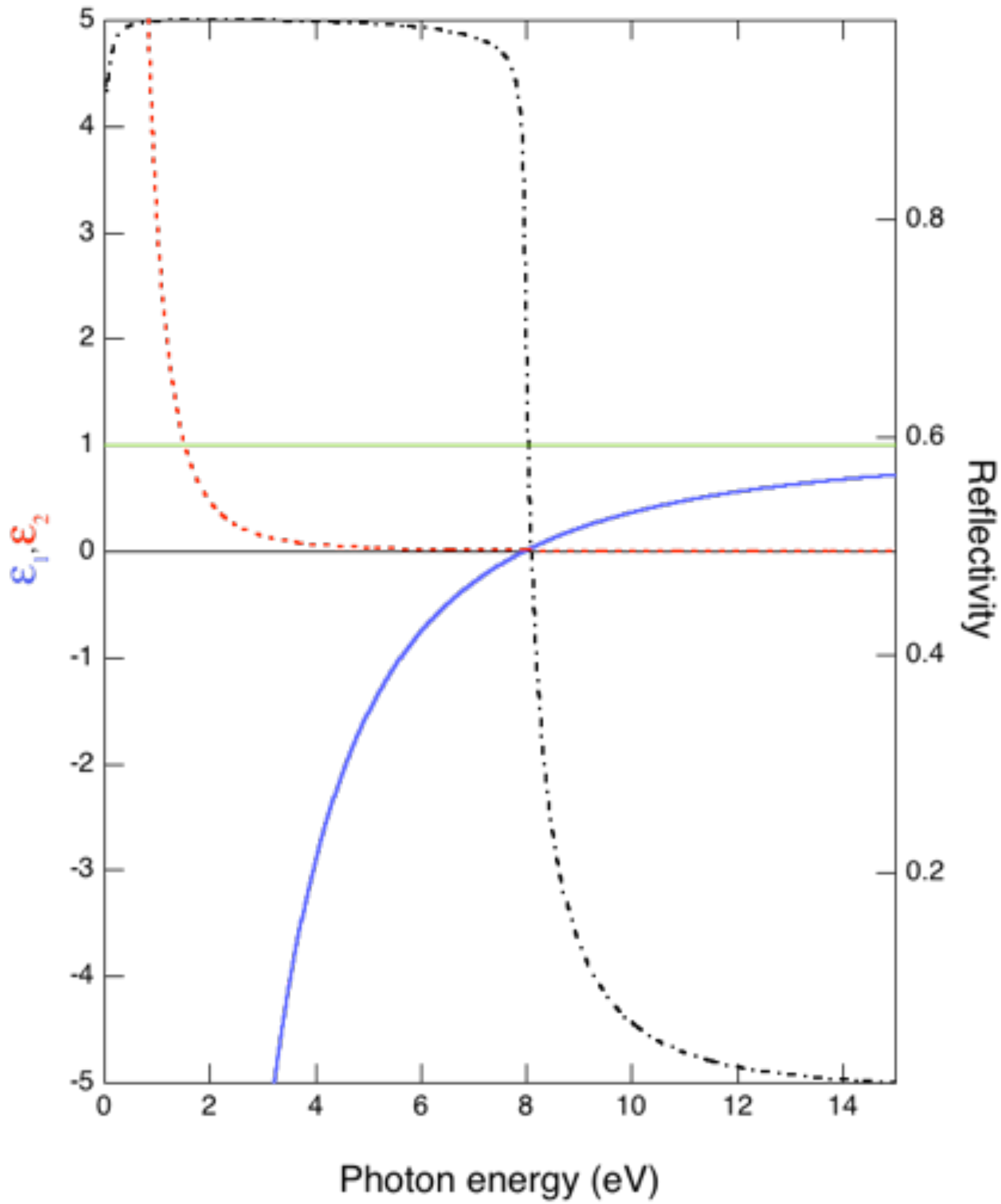

Macroscopic dielectric theory

At frequencies much higher than the plasma frequency we get:

$$\epsilon_1 \cong 1 - \frac{\omega_p^2}{\omega^2}$$
$$\epsilon_2 \cong \frac{\omega_p^2 \gamma}{\omega^3}$$

which represent the behaviour of any material at high energy.
If we now consider the KK relation:

$$\epsilon_1(\omega) = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \epsilon_2(\omega')}{\omega'^2 - \omega^2} d\omega'$$

At high energy we can neglect ω'^2 in the integrand denominator because ϵ_2 decreases very rapidly (as $1/\omega^3$, superconvergence theorem) and therefore:

$$\epsilon_1(\omega)_{\omega \to \infty} \cong 1 - \frac{2}{\omega^2 \pi} \int_0^\infty \omega' \epsilon_2(\omega') d\omega'$$

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$$\epsilon_1(\omega)_{\omega\to\infty} \cong 1 - \frac{2}{\omega^2 \pi} \int_0^\infty \omega' \epsilon_2(\omega') d\omega'$$

Macroscopic dielectric theory

so we get the sum rule:

$$\int_0^{\infty} \omega' \epsilon_2(\omega') d\omega' = \frac{\pi}{2}\omega_p^2 = \frac{2\pi^2 e^2 N}{m}$$

which relates he dielectric function to the total number of electrons contributing to it. We can define the effective number of electrons contributing to the dielectric function up to a frequency ω_{max}:

$$\int_0^{\omega_{max}} \omega' \epsilon_2(\omega') d\omega' \cong \frac{\pi}{2}\frac{4\pi e^2}{m} N_{eff}$$

Macroscopic dielectric theory

The other KK relation

$$\epsilon_2(\omega) = -\frac{2\omega}{\pi} P\int_0^\infty \frac{\epsilon_1(\omega') - 1}{\omega'^2 - \omega^2} d\omega'$$

at high frequency becomes

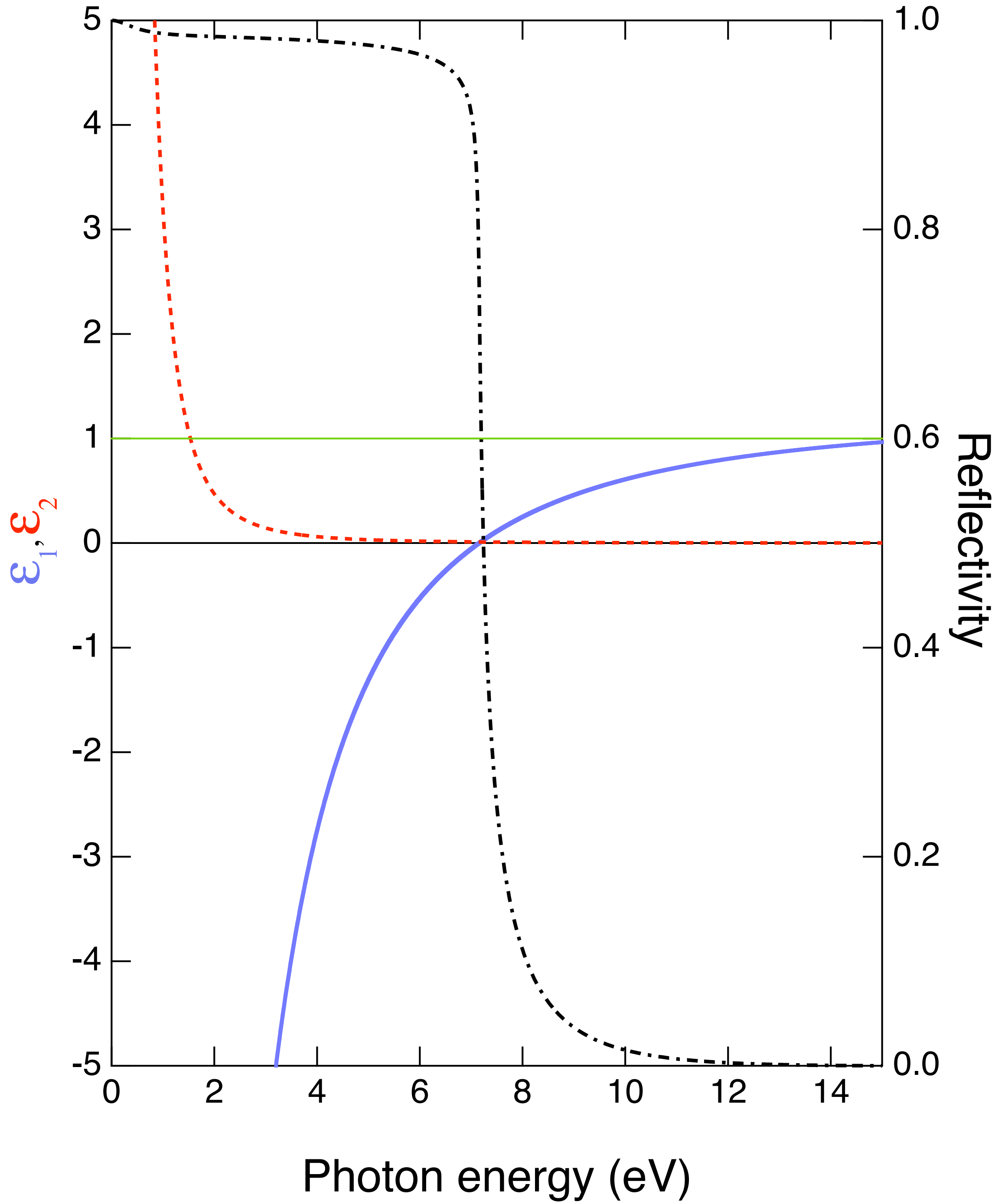
$$\epsilon_2\left(\omega\right)_{\omega\to\infty} = \frac{2}{\pi\omega}\int_0^\infty \left(\epsilon_1\left(\omega'\right) - 1\right) d\omega'$$

and by comparison with the expression of the dielectric function at high energy we get:

$$\int_0^\infty \left(\epsilon_1\left(\omega\right) - 1\right) d\omega = 0$$

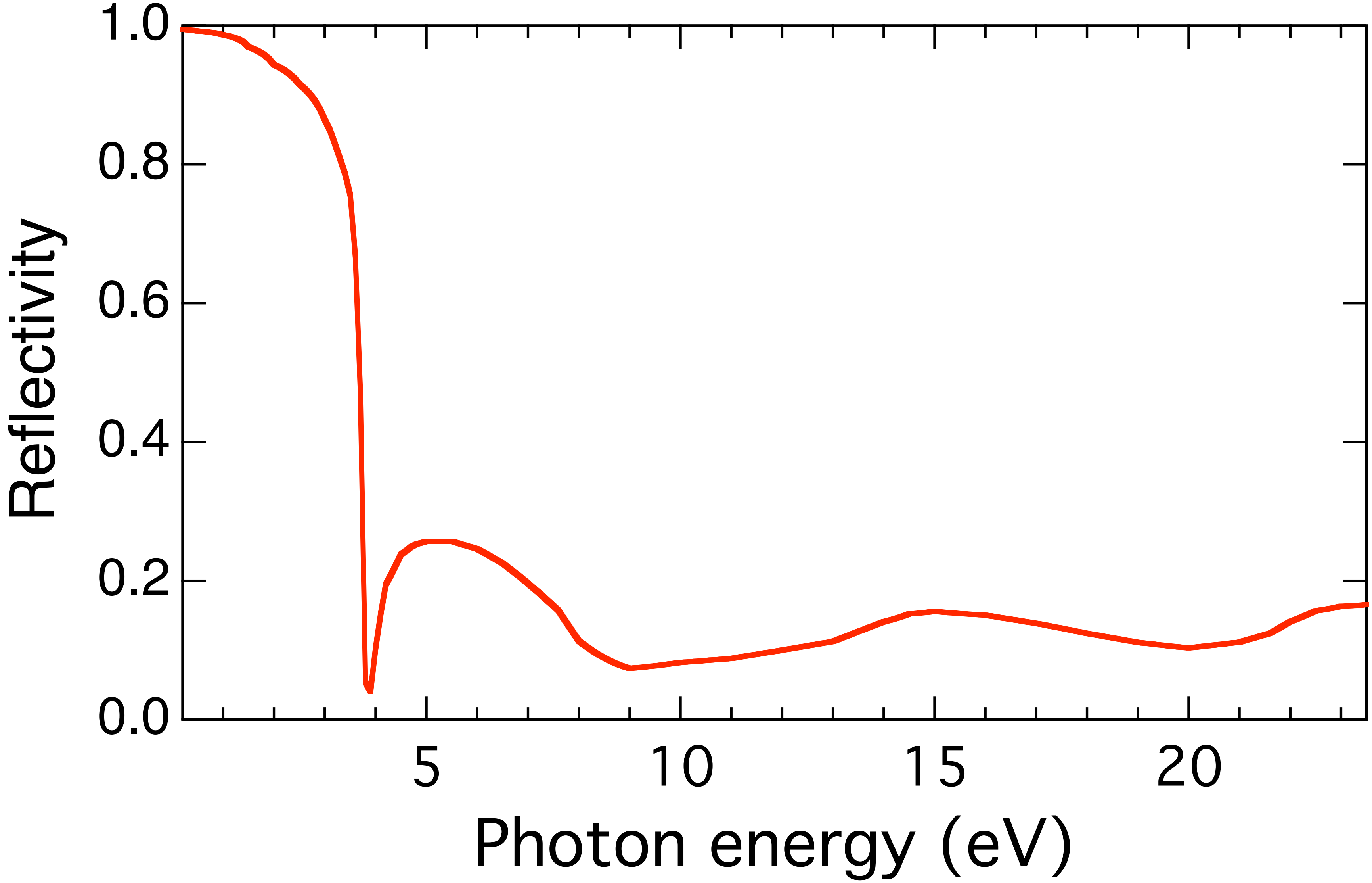
Macroscopic dielectric theory

Free electron metal with $\hbar\omega_p$=8eV and $\hbar\gamma$=0.5 eV



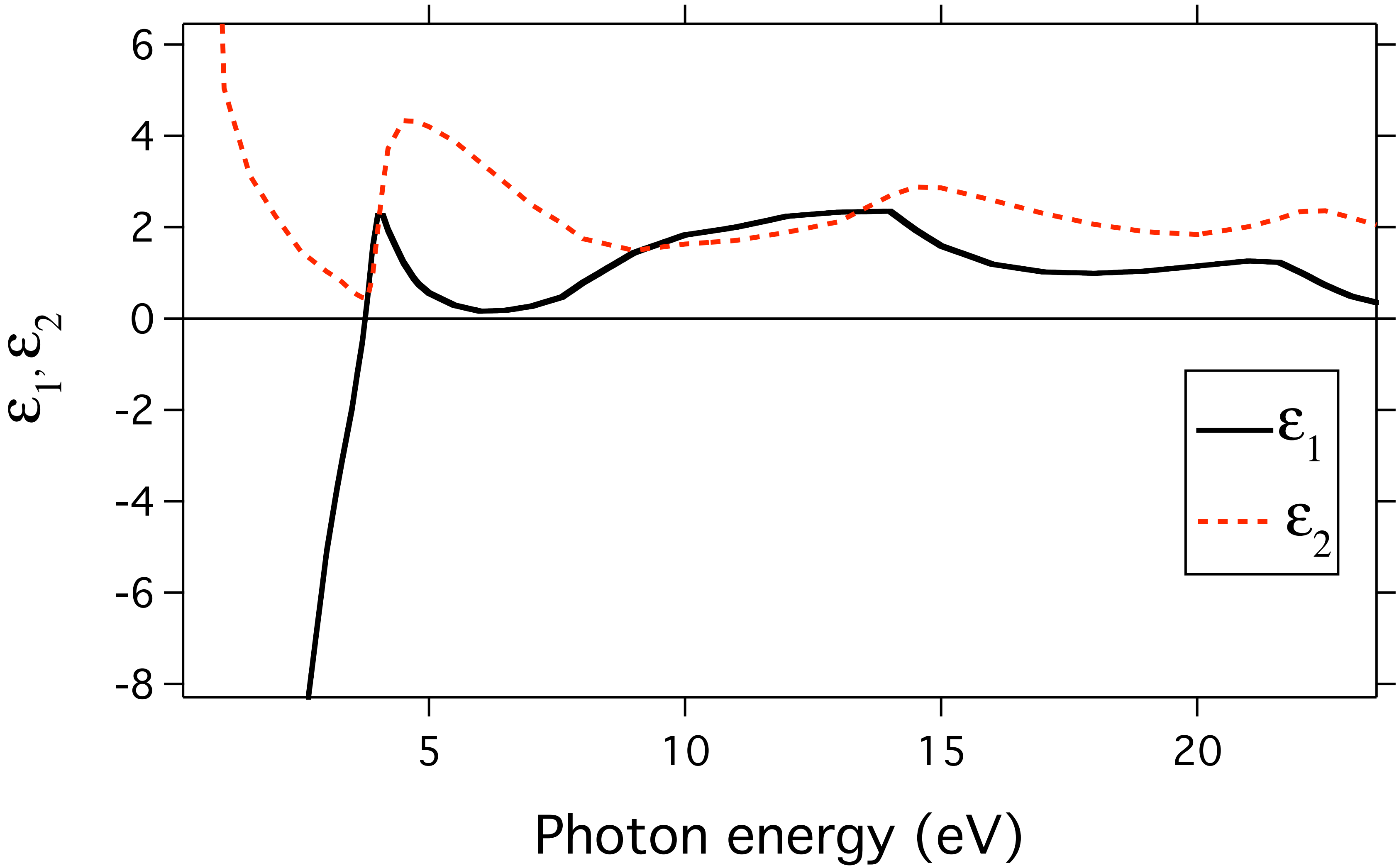
Macroscopic dielectric theory

Normal incidence Ag experimental Reflectivity



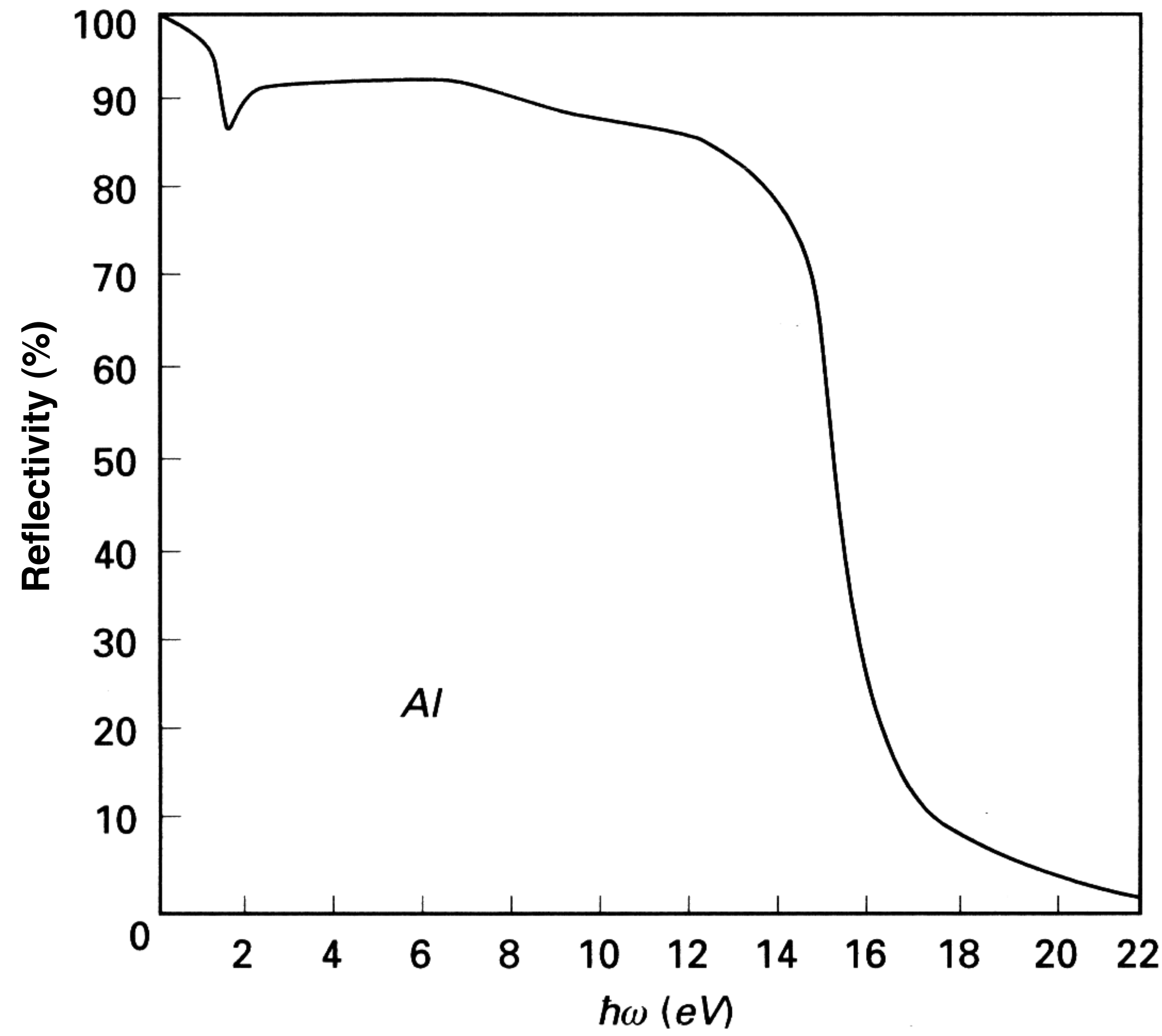
Macroscopic dielectric theory

Experimental dielectric function of Ag:



Macroscopic dielectric theory

Experimental
Reflectivity of Al



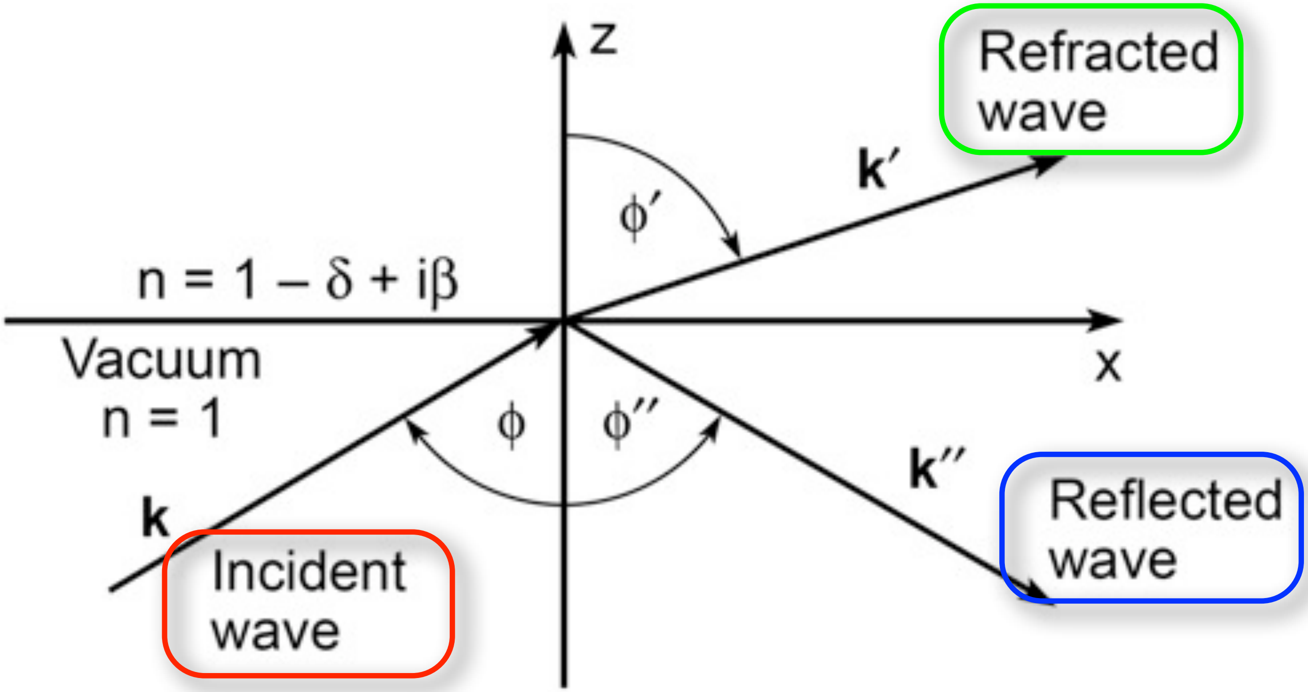
Macroscopic dielectric theory

Reflection and refraction at an arbitrary angle

$$\vec{E} = \vec{E}_0 e^{-i\left(\omega t - \vec{k}\cdot\vec{r}\right)}$$

$$\vec{E'} = \vec{E'_0} e^{-i\left(\omega t - \vec{k'}\cdot\vec{r}\right)}$$

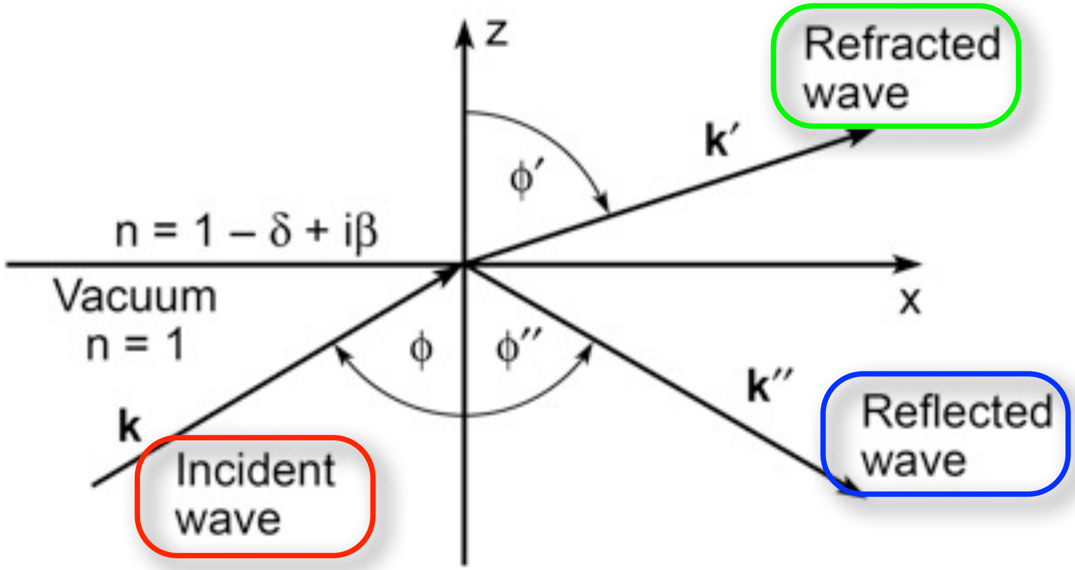
$$\vec{E''} = \vec{E''_0} e^{-i\left(\omega t - \vec{k''}\cdot\vec{r}\right)}$$



All waves have the same frequency, ω, and $\left|\vec{k}\right| = \left|\vec{k''}\right| = \frac{\omega}{c}$

The refracted wave has phase velocity $v_\phi = \frac{\omega}{k'} = \frac{c}{n} \Rightarrow k' = \left|\vec{k'}\right| = \frac{\omega}{c}(1 - \delta + i\beta)$

Macroscopic dielectric theory



Kinematic boundary conditions:

at the boundary (z=0): $\vec{k} \cdot \vec{r_0} = \vec{k'} \cdot \vec{r_0} = \vec{k''} \cdot \vec{r_0}$

nothing occurs along x $k_x = k'_x = k''_x$

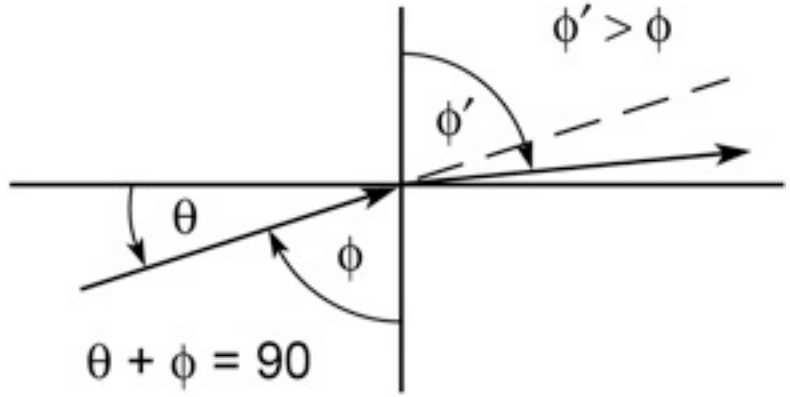
so along z
$$|k_z| = |k'_z| = |k''_z|$$
$$k \sin\phi = k' \sin\phi' = k'' \sin\phi''$$

therefore: $$\phi = \phi''; \qquad \frac{\sin\phi}{\sin\phi'} = n$$

Macroscopic dielectric theory

If we write the complex index of refraction as

$$\tilde{n} = (1 - \delta + i\beta)$$

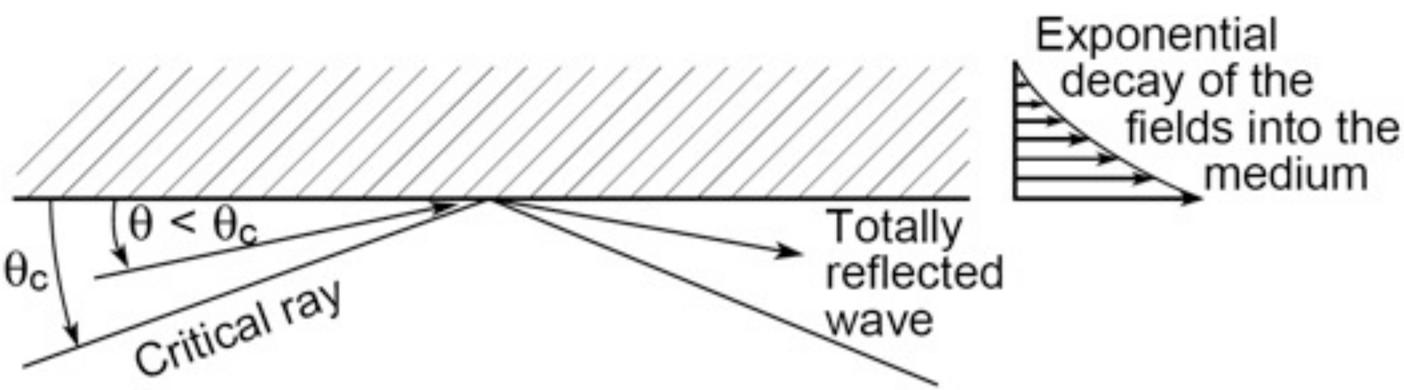


and assume $\beta \rightarrow 0$ then $n \simeq 1-\delta$ and we can have total *external* reflection for angles above the critical angle

$$\phi_c = \arcsin(1 - \delta)$$

or, in terms of the glancing incidence

$$\theta_c = \sqrt{2\delta}$$



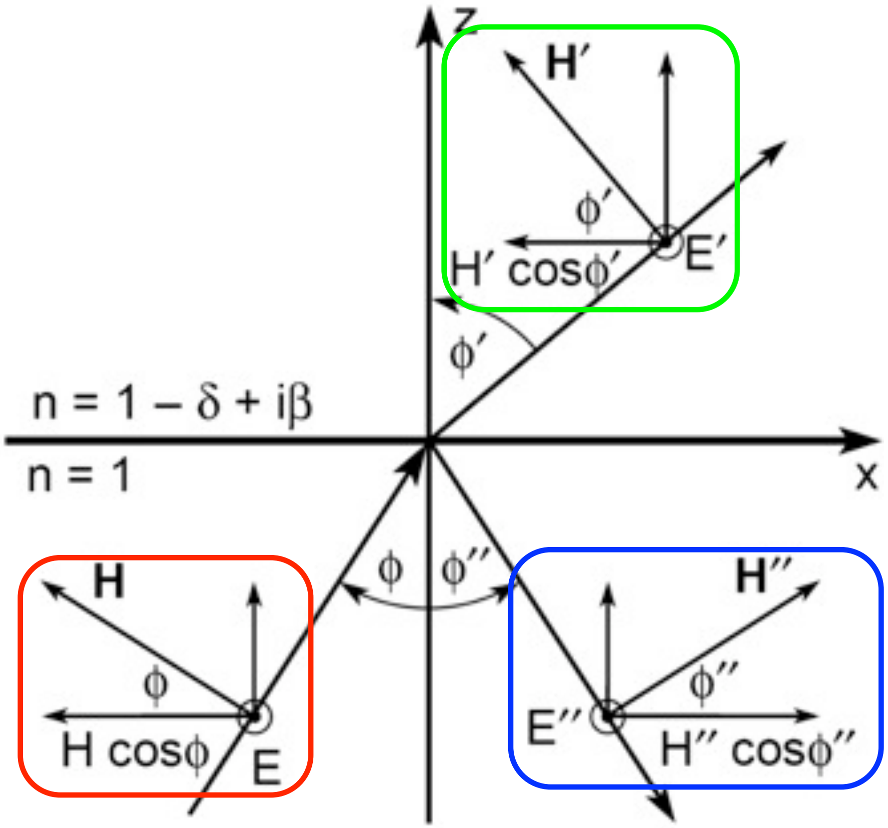
Macroscopic dielectric theory

Dynamic boundary conditions for an ***s* polarized** wave:

$$\vec{E} = \vec{E}_0 e^{-i\left(\omega t - \vec{k}\cdot\vec{r}\right)}$$

$$\vec{E'} = \vec{E'_0} e^{-i\left(\omega t - \vec{k'}\cdot\vec{r}\right)}$$

$$\vec{E''} = \vec{E''_0} e^{-i\left(\omega t - \vec{k''}\cdot\vec{r}\right)}$$

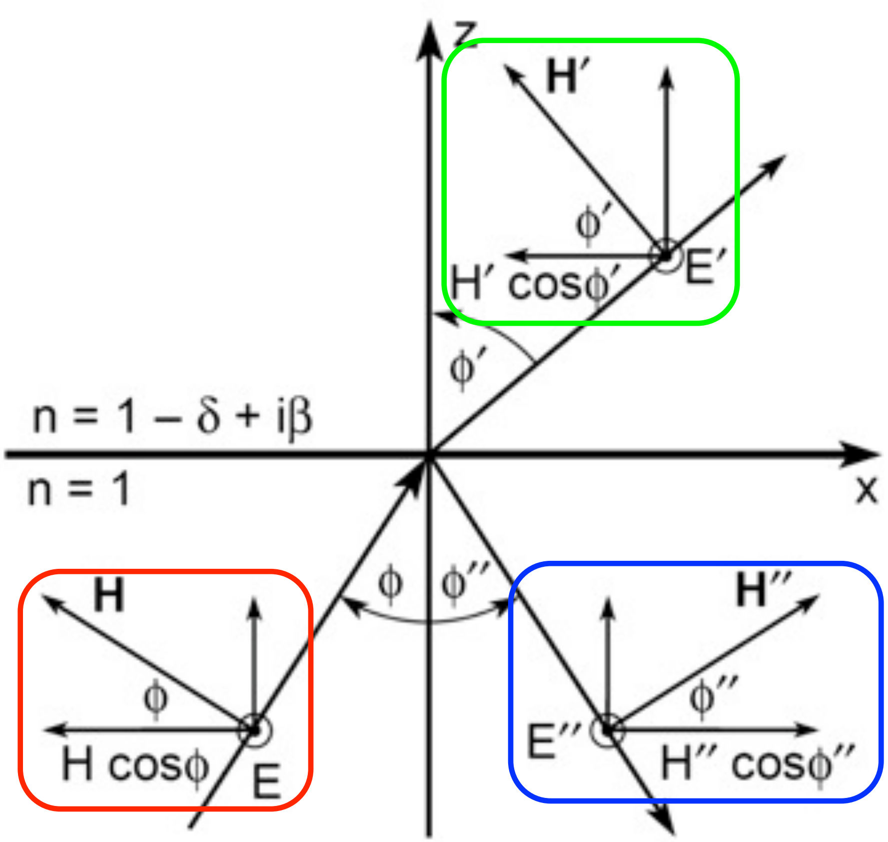


Macroscopic dielectric theory

$$\vec{E} = \vec{E}_0 e^{-i\left(\omega t - \vec{k}\cdot\vec{r}\right)}$$

$$\vec{E}' = \vec{E}'_0 e^{-i\left(\omega t - \vec{k'}\cdot\vec{r}\right)}$$

$$\vec{E}'' = \vec{E}''_0 e^{-i\left(\omega t - \vec{k''}\cdot\vec{r}\right)}$$



Dynamic boundary conditions for an ***s* polarized** wave:

tangential electric and magnetic fields are continuos:

$$E_0 = E'_0 = E''_0$$

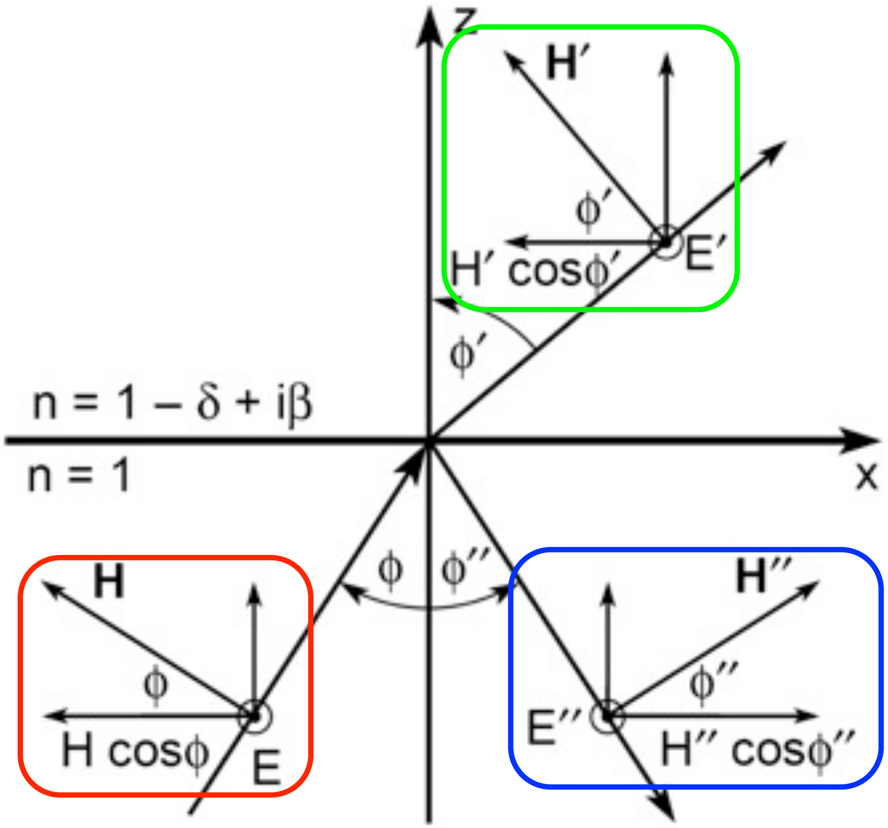
$$H_0 \cos\phi - H''_0 \cos\phi = H'_0 \cos\phi'$$

Macroscopic dielectric theory

$$\vec{E} = \vec{E}_0 e^{-i\left(\omega t - \vec{k}\cdot\vec{r}\right)}$$

$$\vec{E'} = \vec{E'_0} e^{-i\left(\omega t - \vec{k'}\cdot\vec{r}\right)}$$

$$\vec{E''} = \vec{E''_0} e^{-i\left(\omega t - \vec{k''}\cdot\vec{r}\right)}$$



Dynamic boundary conditions for an *s* polarized wave:

tangential electric and magnetic fields are continuos:

$$E_0 = E_0' = E_0''$$

$$H_0 \cos\phi - H_0'' \cos\phi = H_0' \cos\phi'$$

since

$$\vec{H}\left(\vec{r}, t\right) = \tilde{n}\frac{\vec{k}}{k} \times \vec{E}\left(\vec{r}, t\right)$$

we obtain

$$\left(E_0 - E_0''\right)\cos\phi = \tilde{n}E_0' \cos\phi'$$

Macroscopic dielectric theory

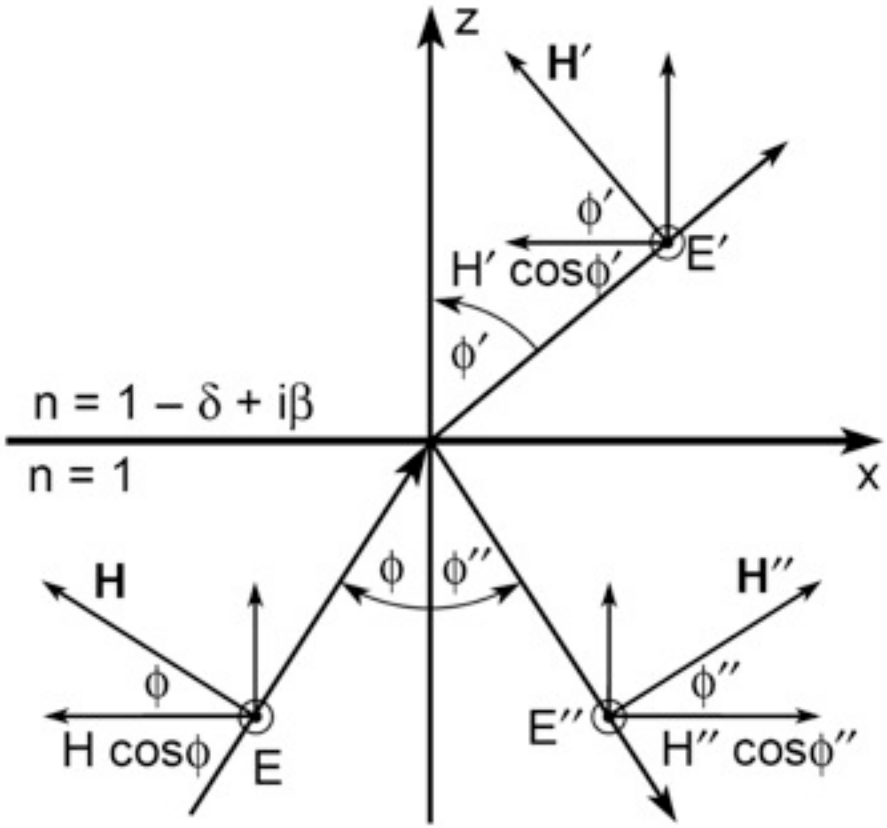
For an *s* polarized wave:

$$\frac{E_0'}{E_0} = \frac{2\cos\phi}{\cos\phi + \sqrt{n^2 - \sin^2\phi}}$$

$$\frac{E_0''}{E_0} = \frac{\cos\phi - \sqrt{n^2 - \sin^2\phi}}{\cos\phi + \sqrt{n^2 - \sin^2\phi}}$$

so the reflectivity is

$$R_s = \frac{\left|\cos\phi - \sqrt{n^2 - \sin^2\phi}\right|^2}{\left|\cos\phi + \sqrt{n^2 - \sin^2\phi}\right|^2}$$

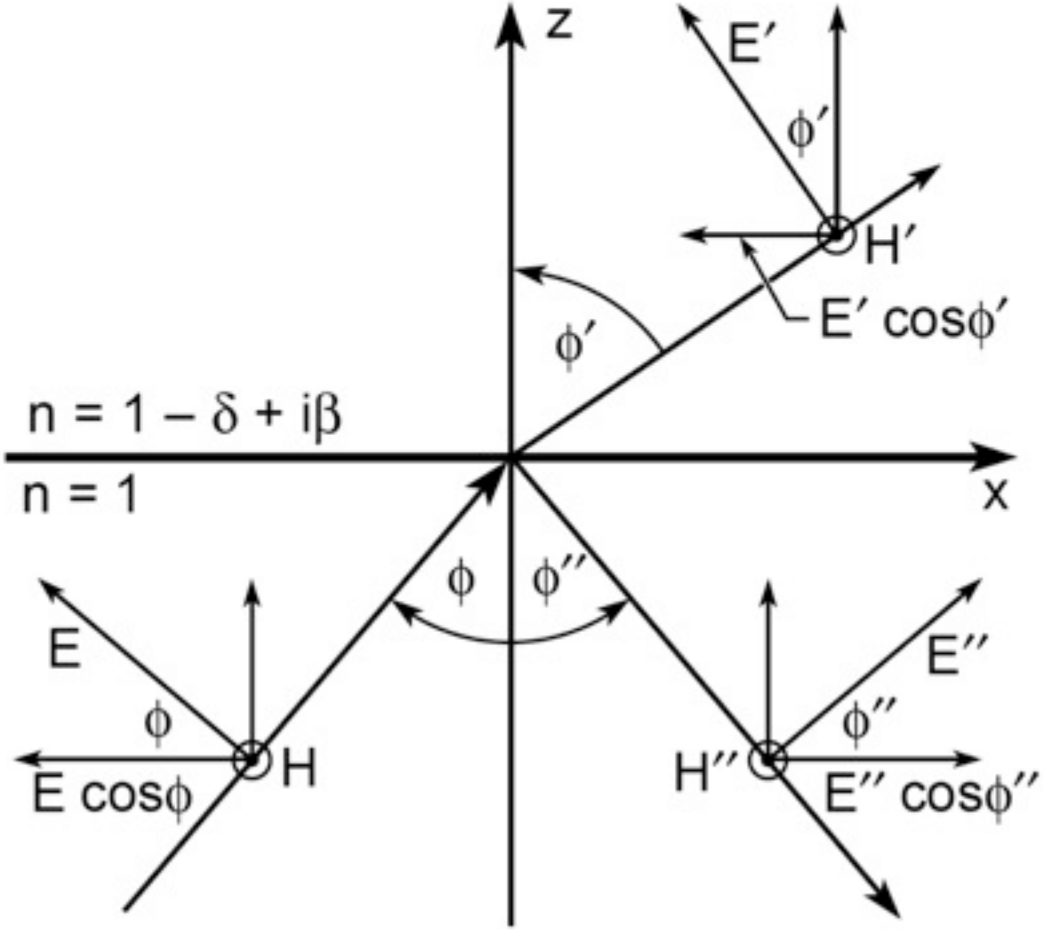


Macroscopic dielectric theory

For a *p* polarized wave:

$$\frac{E_0'}{E_0} = \frac{2n\cos\phi}{n^2\cos\phi + \sqrt{n^2 - \sin^2\phi}}$$

$$\frac{E_0''}{E_0} = \frac{n^2\cos\phi - \sqrt{n^2 - \sin^2\phi}}{n^2\cos\phi + \sqrt{n^2 - \sin^2\phi}}$$

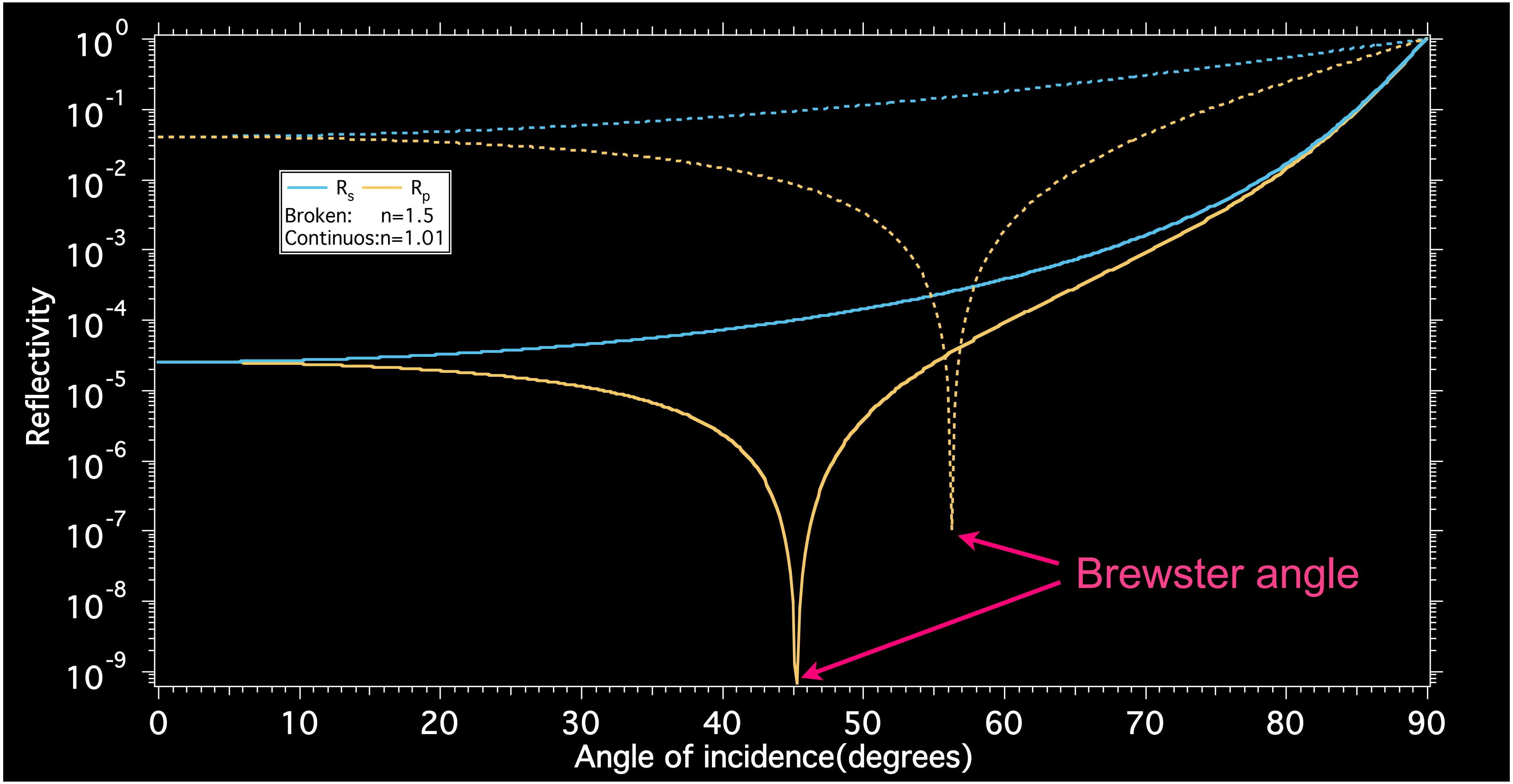


so the reflectivity is

$$R_p = \frac{\left|n^2\cos\phi - \sqrt{n^2 - \sin^2\phi}\right|^2}{\left|n^2\cos\phi + \sqrt{n^2 - \sin^2\phi}\right|^2}$$

Macroscopic dielectric theory

Reflectivity as a function of ϕ and n

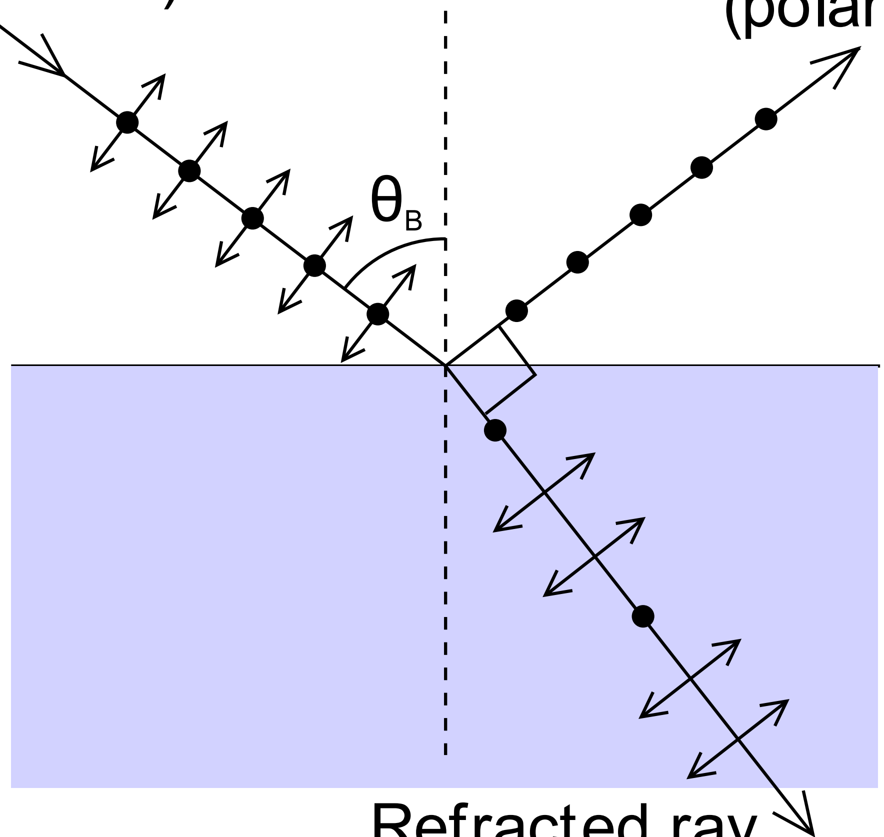


Brewster's angle

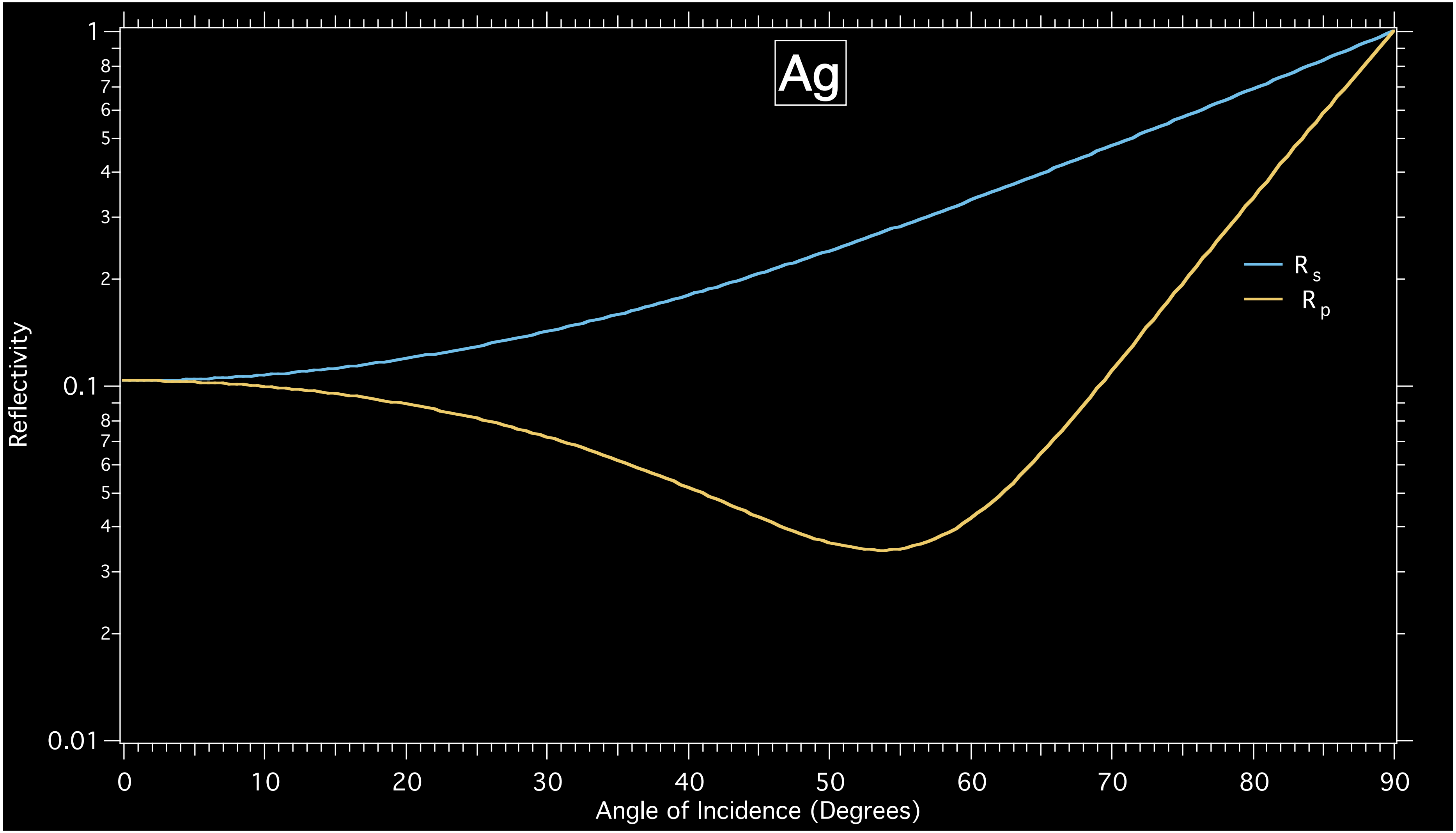
When the reflected and refracted waves form an angle of 90° i.e. when:

$$\theta_b = \arctan\left(\frac{n_2}{n_1}\right)$$

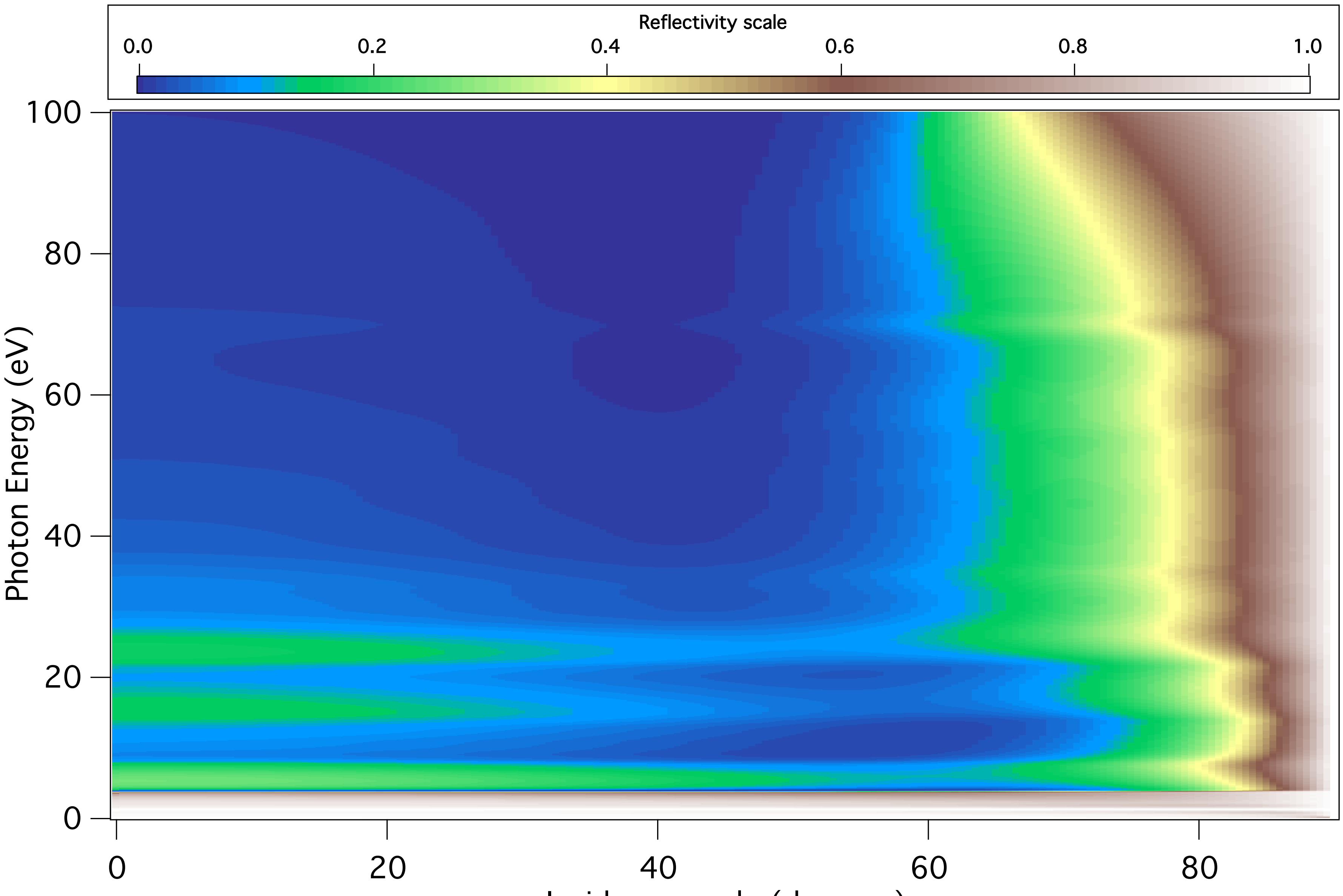
the reflected wave is 100% s polarized (if no absorption occurs!)



Ag s- and p- polarized reflectivity at $\hbar\omega$=20eV

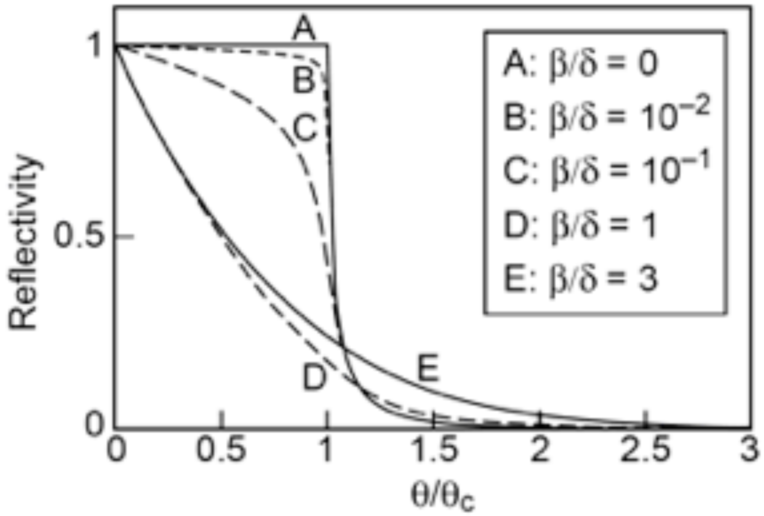


Experimental reflectivity of Ag for p-polarized light



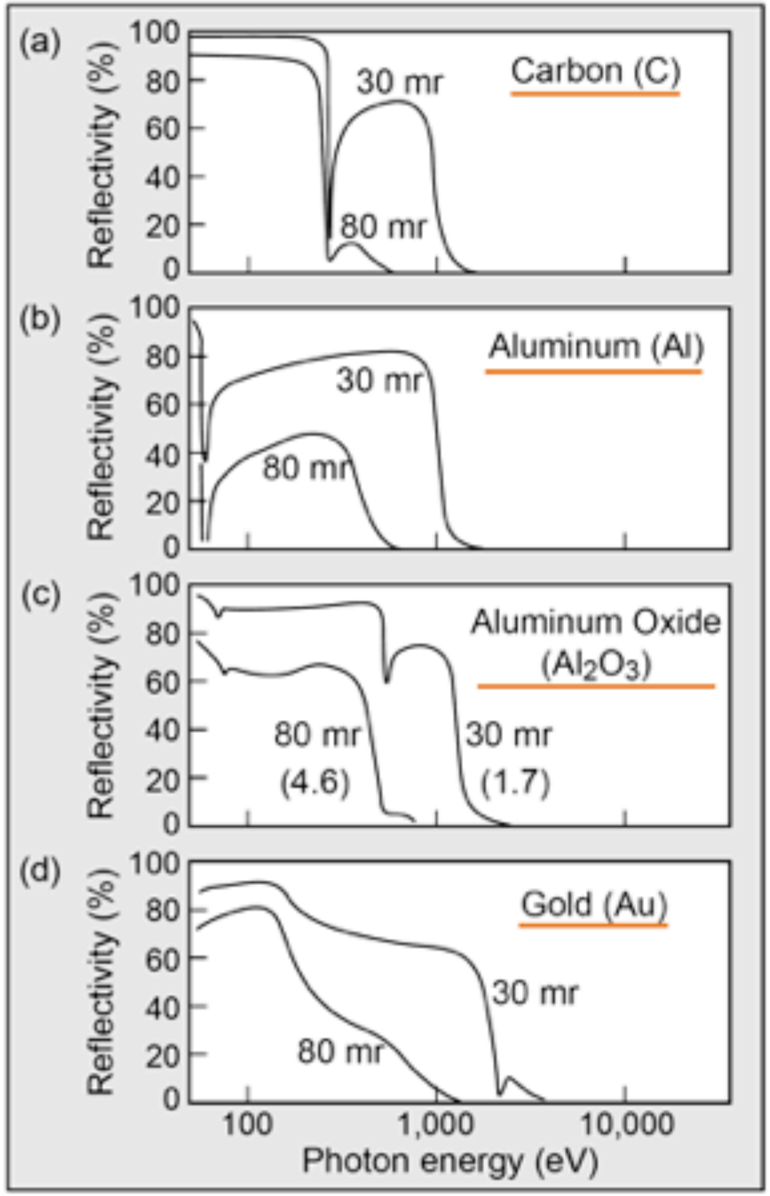
Macroscopic dielectric theory

Glancing incidence reflection
as a function of β/δ



- finite β/δ rounds the sharp angular dependence
- cutoff angle and absorption edges can enhance the sharpness
- note the effects of oxide layers and surface contamination

. . . for real materials



(Henke, Gullikson, Davis)