# Generation of Synchrotron Radiation

# Characteristics of synchrotron radiation

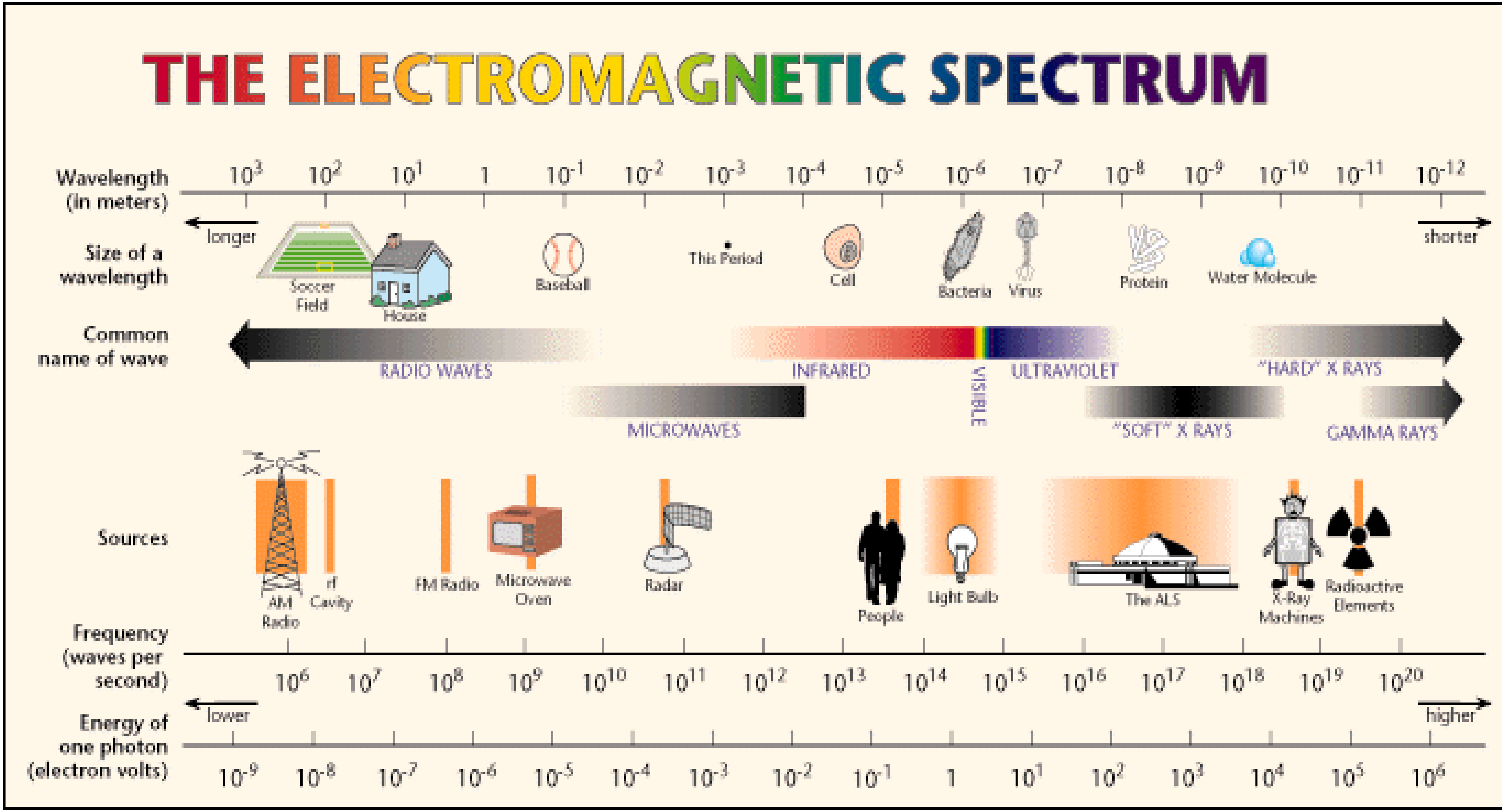
Broad Spectrum

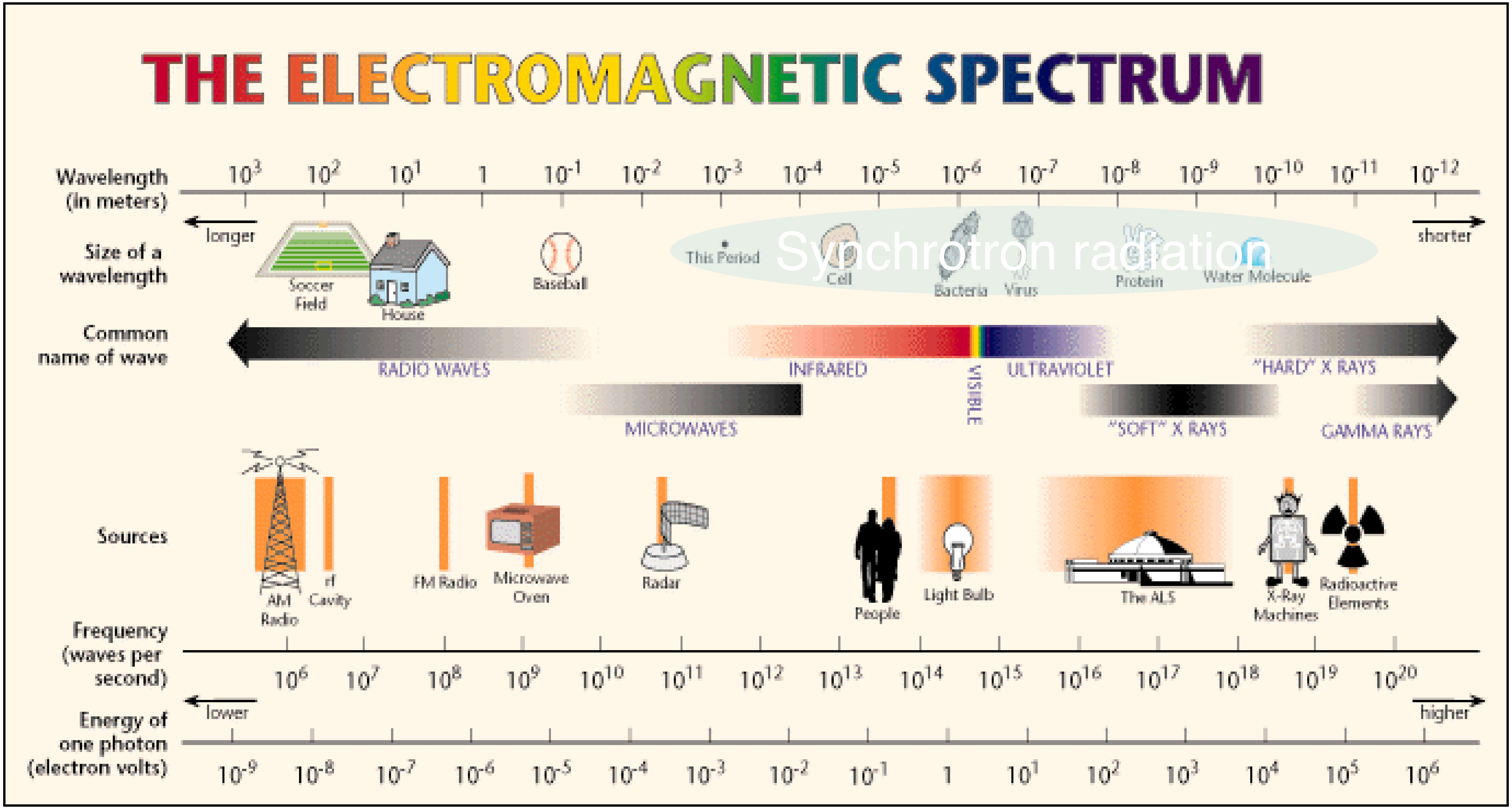
High Flux

Polarisation

Brightness: small divergence, small source size

Time Structure


THE ELECTROMAGNETIC SPECTRUM
Wavelength (in meters)
10^3 10^2 10^1 1 10^-1 10^-2 10^-3 10^-4 10^-5 10^-6 10^-7 10^-8 10^-9 10^-10 10^-11 10^-12
longer
shorter
Size of a wavelength
Soccer Field
House
Baseball
This Period
Cell
Bacteria
Virus
Protein
Water Molecule
Common name of wave
RADIO WAVES
INFRARED
VISIBLE
ULTRAVIOLET
"HARD" X RAYS
MICROWAVES
"SOFT" X RAYS
GAMMA RAYS
Sources
AM Radio
rf Cavity
FM Radio
Microwave Oven
Radar
People
Light Bulb
The ALS
X-Ray Machines
Radioactive Elements
Frequency (waves per second)
10^6 10^7 10^8 10^9 10^10 10^11 10^12 10^13 10^14 10^15 10^16 10^17 10^18 10^19 10^20
lower
higher
Energy of one photon (electron volts)
10^-9 10^-8 10^-7 10^-6 10^-5 10^-4 10^-3 10^-2 10^-1 1 10^1 10^2 10^3 10^4 10^5 10^6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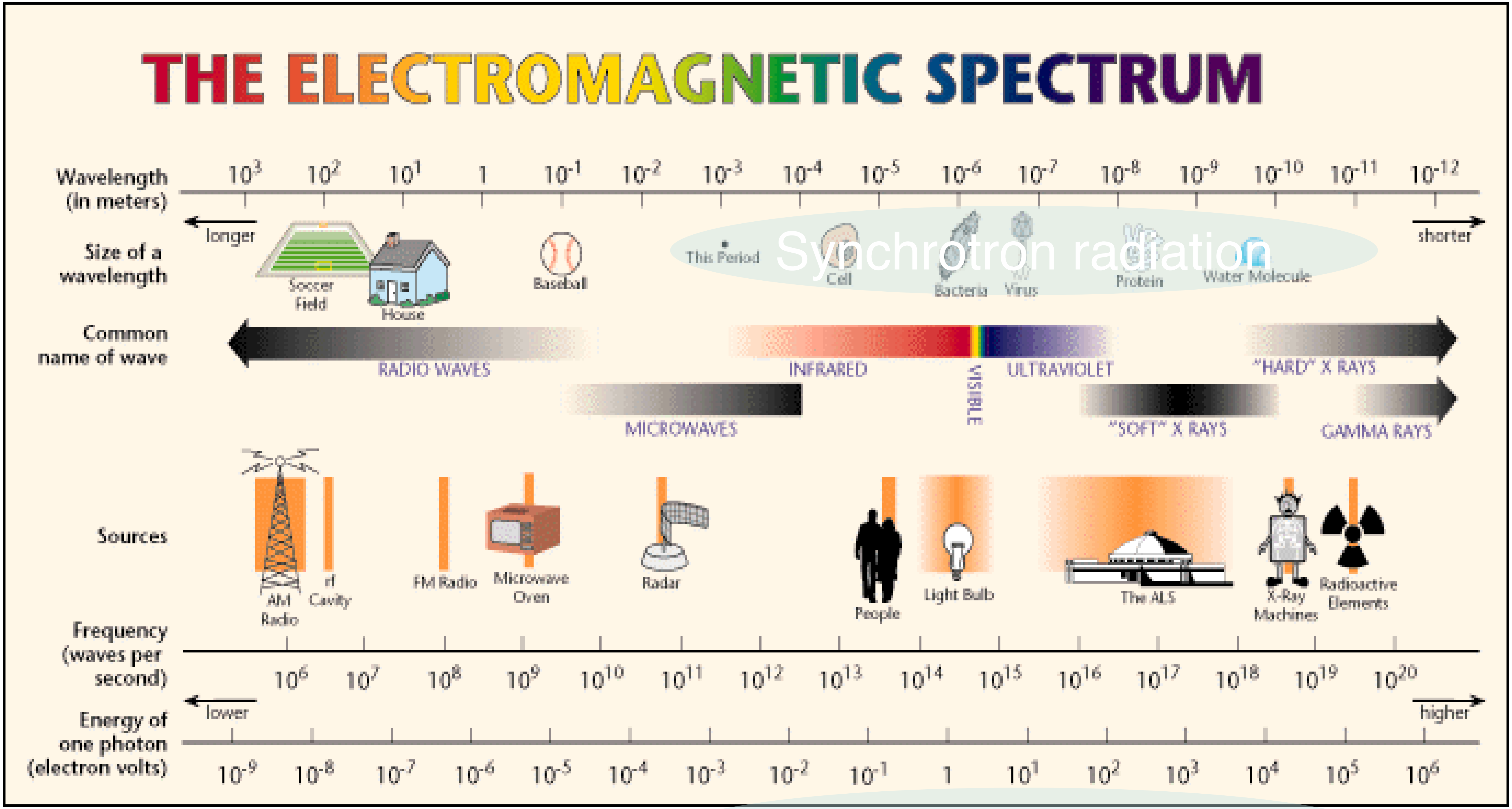


Synchrotron radiation

Synchrotron radiation

# A synchrotron light source

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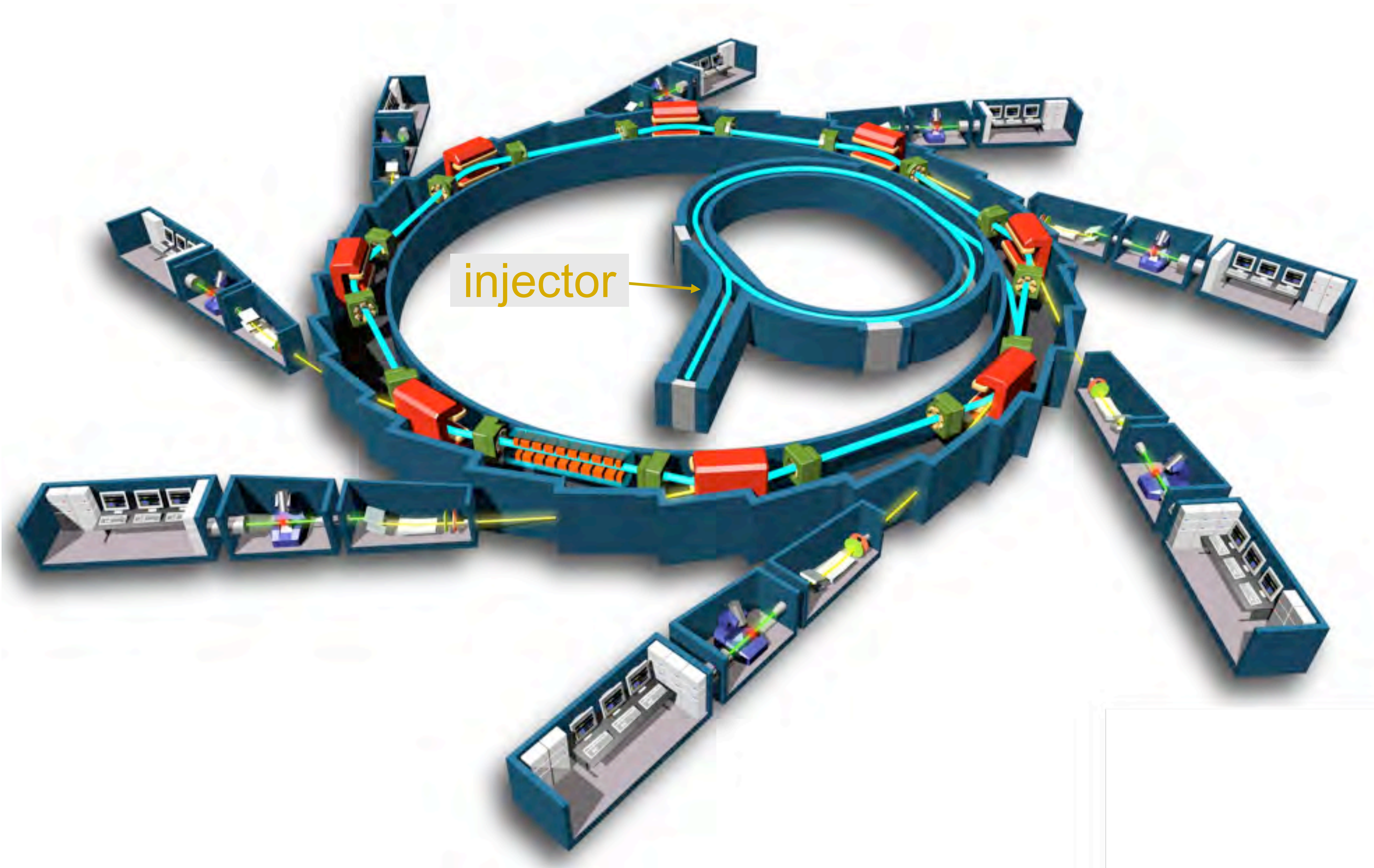

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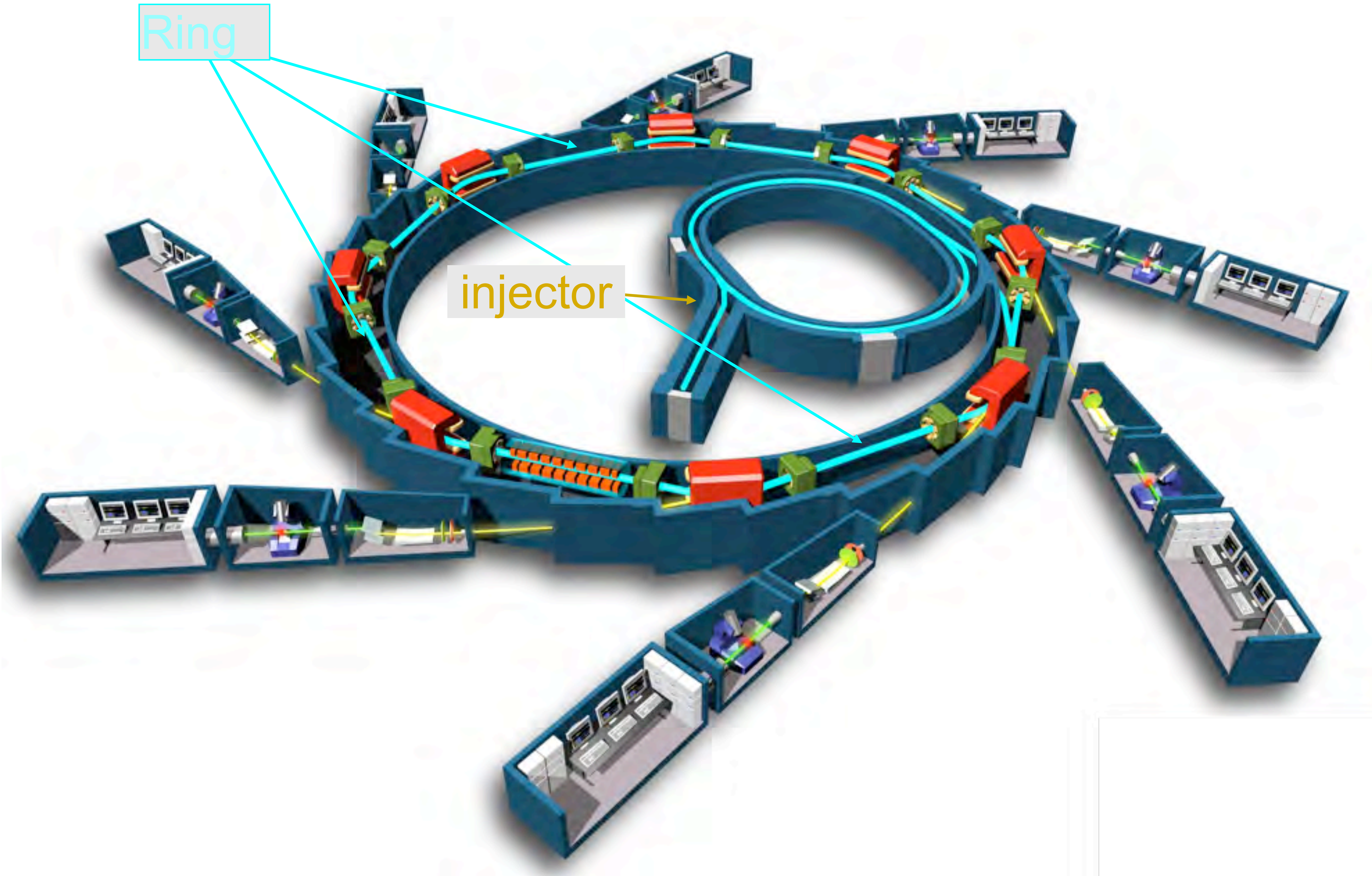

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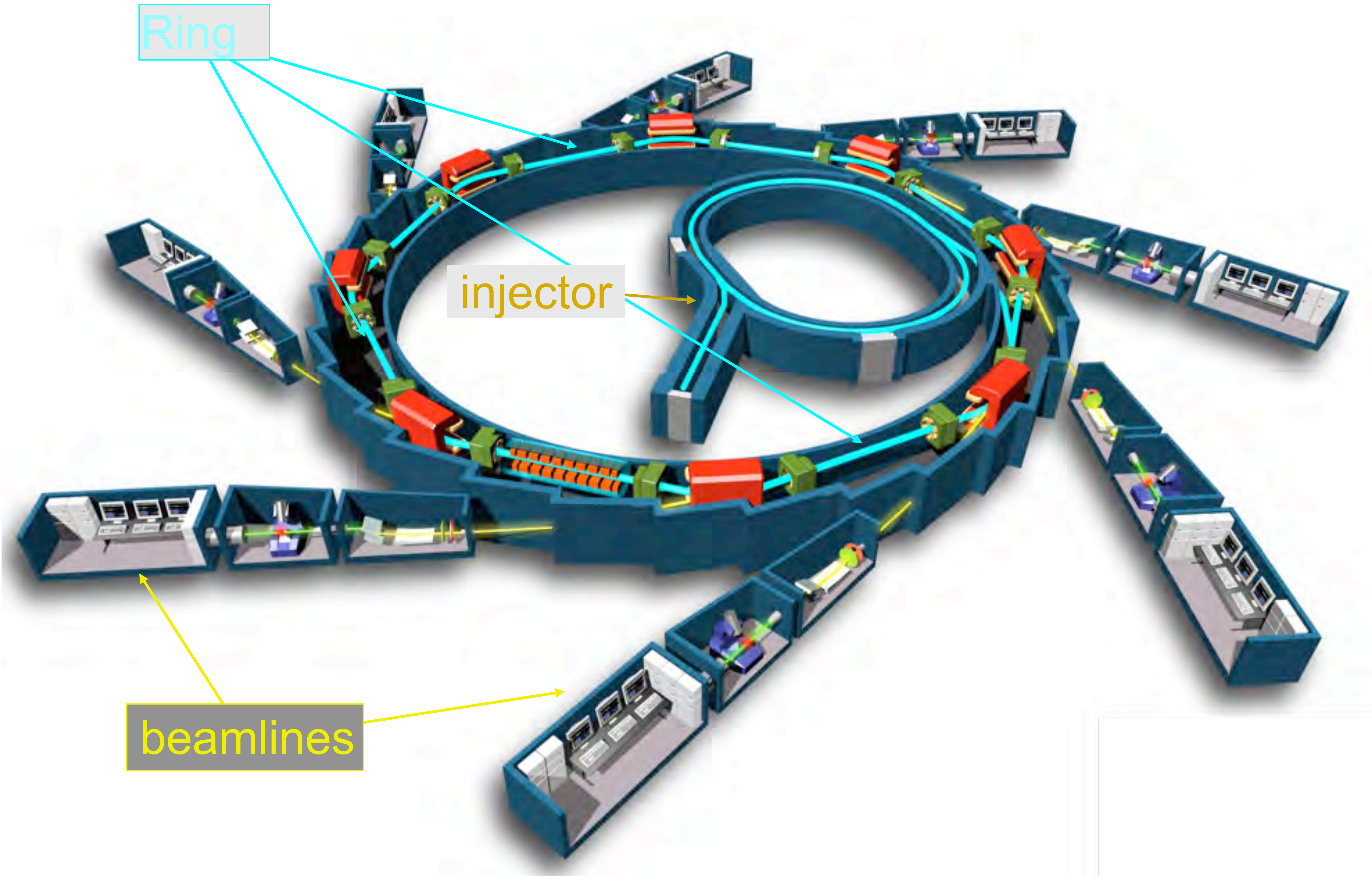

# First observation of synchrotron radiation

PHYSICAL REVIEW VOLUME 74, NUMBER 1 JULY 1, 1948

## Radiation from Electrons Accelerated in a Synchrotron

F. R. ELDER, R. V. LANGMUIR, AND H. C. POLLOCK
*General Electric Company, Schenectady, New York*

(Received March 15, 1948)

High energy electrons subjected to large radial accelerations radiate considerable energy in the optical spectrum. The distribution of energy in the light from a synchrotron beam has been measured and compared with theory at several electron energies up to 80 Mev. The results indicate reasonable agreement with theory. Measurement of total light output allowed an estimate of electron current in the beam. High speed photography of the light permitted observation of the size and motion of the beam within the accelerator tube.

# First observation of synchrotron radiation

Professor J. S. Schwinger of Harvard has calculated the distribution of the energy radiated, and has kindly sent us his results (expressions (1) through (4)).

For an electron of constant energy

$$P(\omega)d\omega = (3\sqrt{3}/4\pi)\omega_0(e^2/R)(E/mc^2)^4 \times\left[\int_{\omega/\omega_c}^{\infty} K_{5/3}(x)dx\right](\omega/\omega_c)d\omega, \quad (1)$$

where $P(\omega)d\omega$ is the power radiated by one electron at the circular frequency $\omega$ in the range $d\omega$. $R$ is the radius of the orbit in cm; $\omega_0$ the angular velocity of the electron, $V/R$; $e$ the electron charge in e.s.u.; $E$ the total electron energy; and $K_{5/3}$ a cylinder function as defined in Watson's treatise on Bessel Functions. $\omega_c = \frac{3}{2}\omega_0(E/mc^2)^3$. $\omega_c$ is a critical frequency which roughly measures the upper limit of the spectrum. The expression for the total power ra-

# .... and the theory

# .... and the theory

PHYSICAL REVIEW VOLUME 75, NUMBER 12 JUNE 15, 1949

## On the Classical Radiation of Accelerated Electrons

JULIAN SCHWINGER
*Harvard University, Cambridge, Massachusetts*

(Received March 8, 1949)

This paper is concerned with the properties of the radiation from a high energy accelerated electron, as recently observed in the General Electric synchrotron. An elementary derivation of the total rate of radiation is first presented, based on Larmor's formula for a slowly moving electron, and arguments of relativistic invariance. We then construct an expression for the instantaneous power radiated by an electron moving along an arbitrary, prescribed path. By casting this result into various forms, one obtains the angular distribution, the spectral distribution, or the combined angular and spectral distributions of the radiation. The method is based on an examination of the rate at which the electron irreversibly transfers energy to the electromagnetic field, as determined by half the difference of retarded and advanced electric field intensities. Formulas are obtained for an arbitrary charge-current distribution and then specialized to a point charge. The total radiated power and its angular distribution are obtained for an arbitrary trajectory. It is found that the direction of motion is a strongly preferred direction of emission at high energies. The spectral distribution of the radiation depends upon the detailed motion over a time interval large compared to the period of the radiation. However, the narrow cone of radiation generated by an energetic electron indicates that only a small part of the trajectory is effective in producing radiation observed in a given direction, which also implies that very high frequencies are emitted. Accordingly, we evaluate the spectral and angular distributions of the high frequency radiation by an energetic electron, in their dependence upon the parameters characterizing the instantaneous orbit. The average spectral distribution, as observed in the synchrotron measurements, is obtained by averaging the electron energy over an acceleration cycle. The entire spectrum emitted by an electron moving with constant speed in a circular path is also discussed. Finally, it is observed that quantum effects will modify the classical results here obtained only at extraordinarily large energies.

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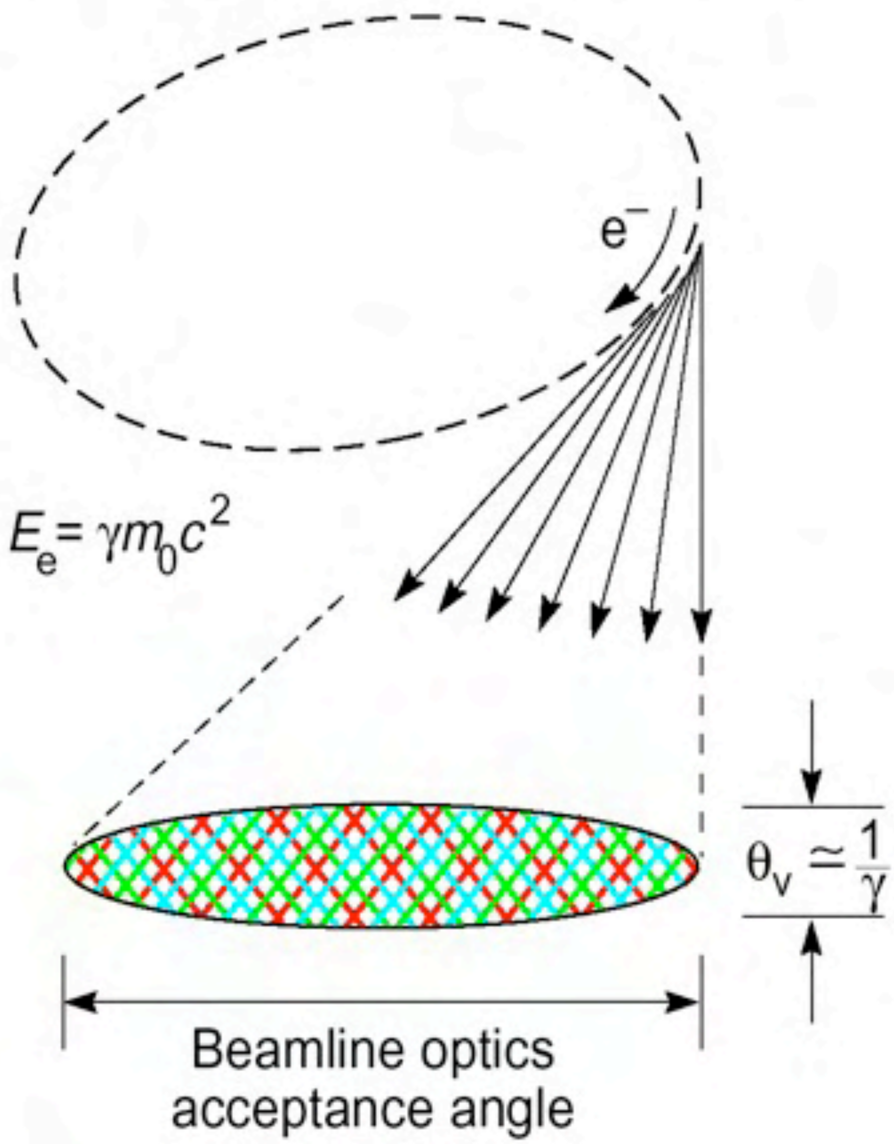
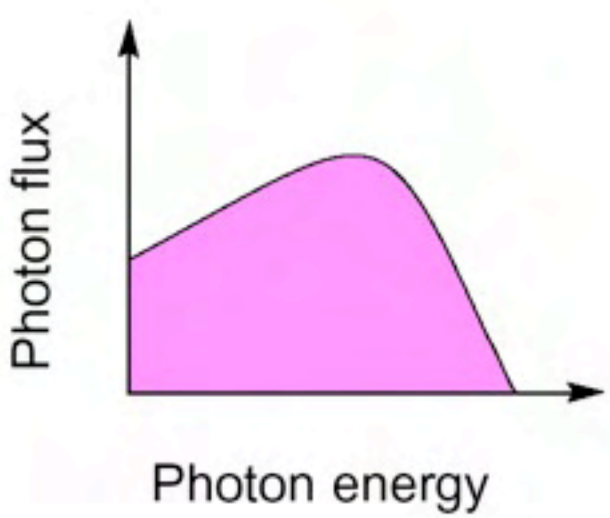
# Bend-Magnet Radiation



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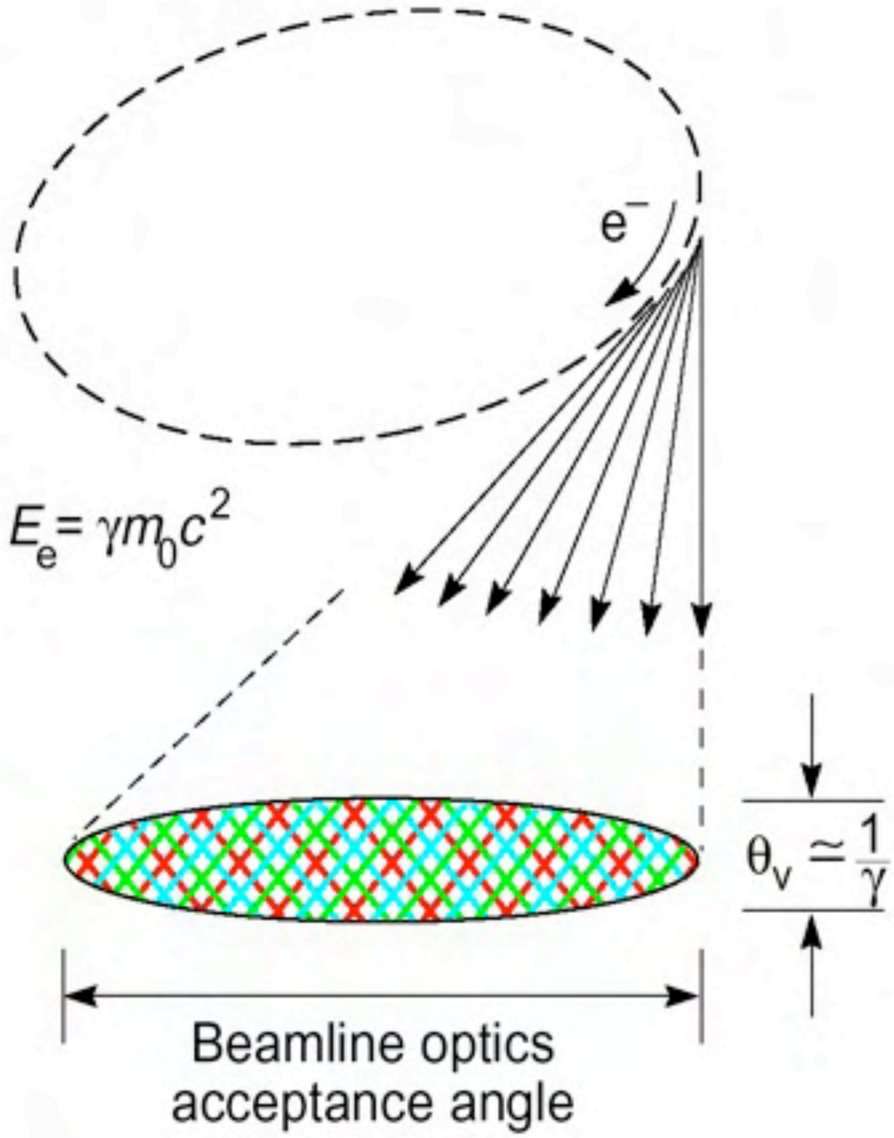
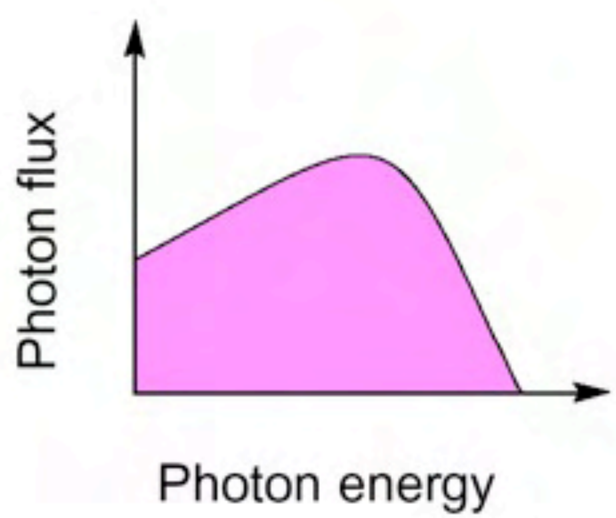
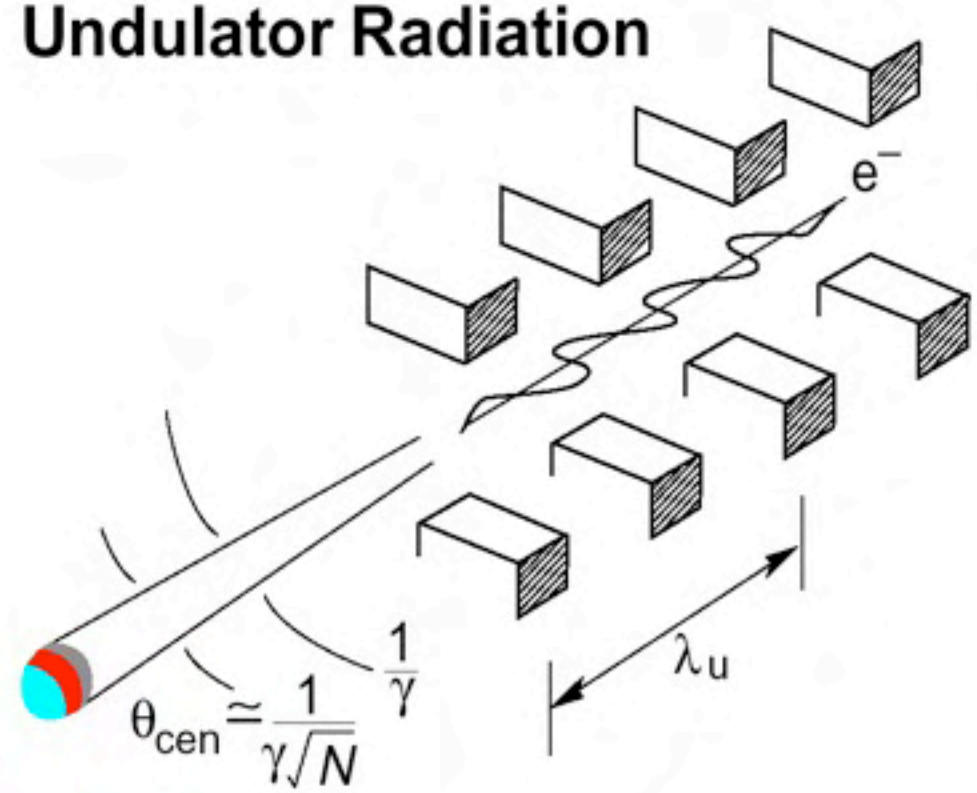



# Undulator Radiation

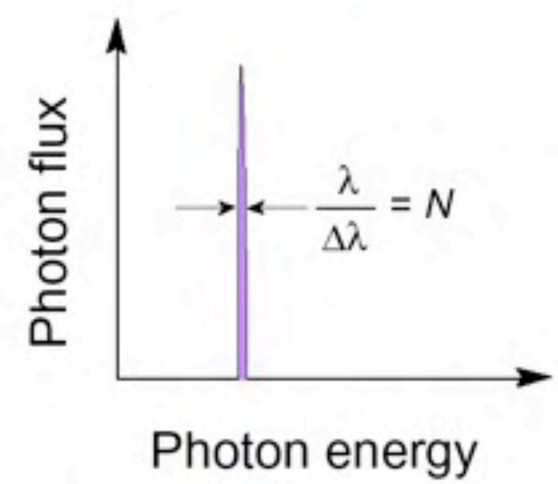



$$\lambda_x = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

In the central radiation cone:

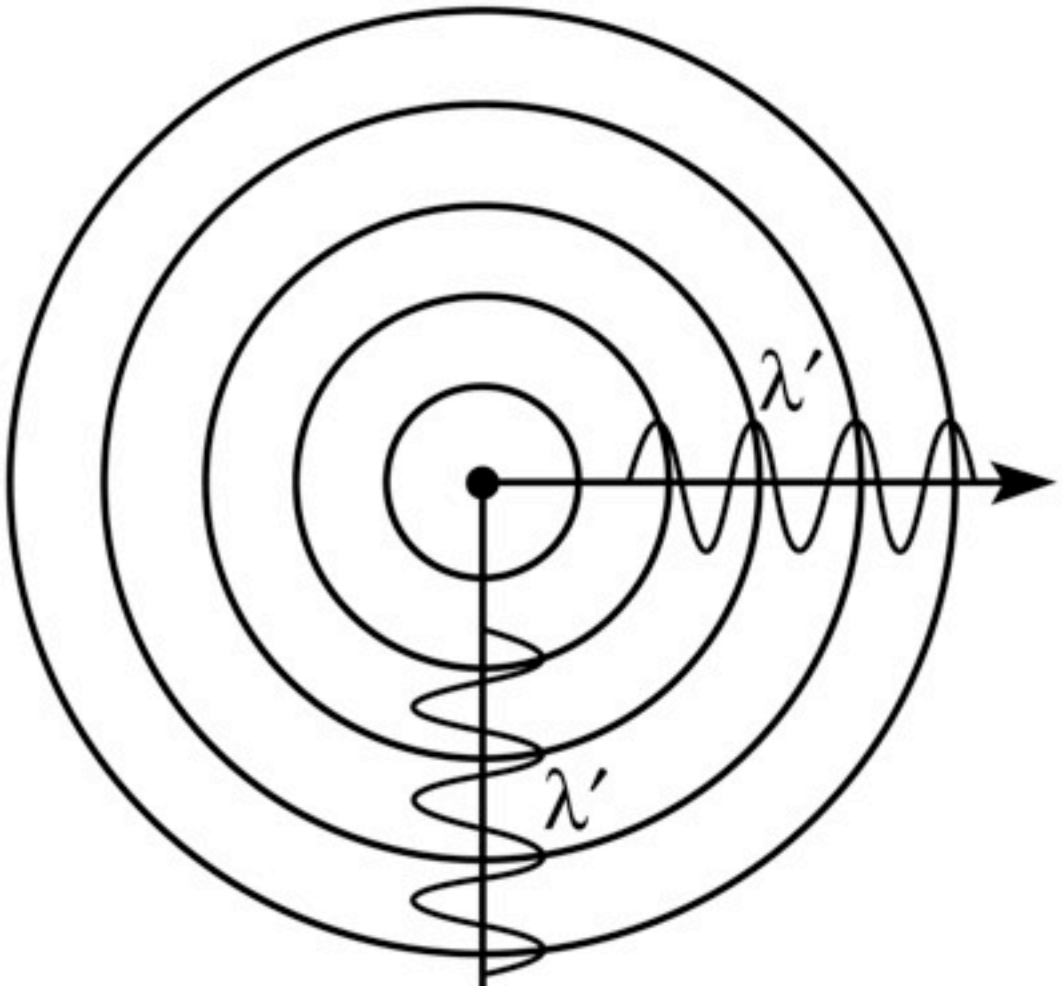
$$\frac{\Delta\omega}{\omega} \simeq \frac{1}{N}$$

$$\theta_{cen} \simeq \frac{1}{\gamma\sqrt{N}}$$



# Doppler shift

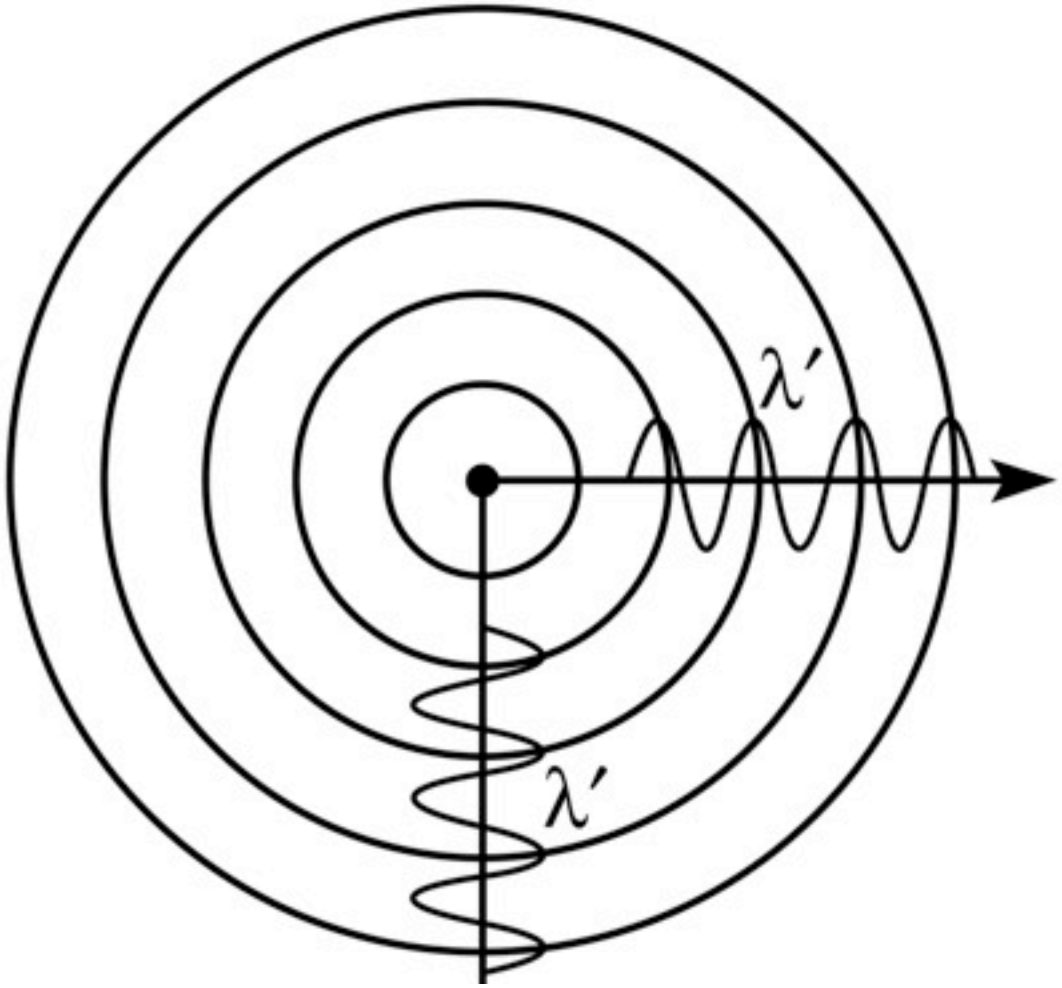
v<<c



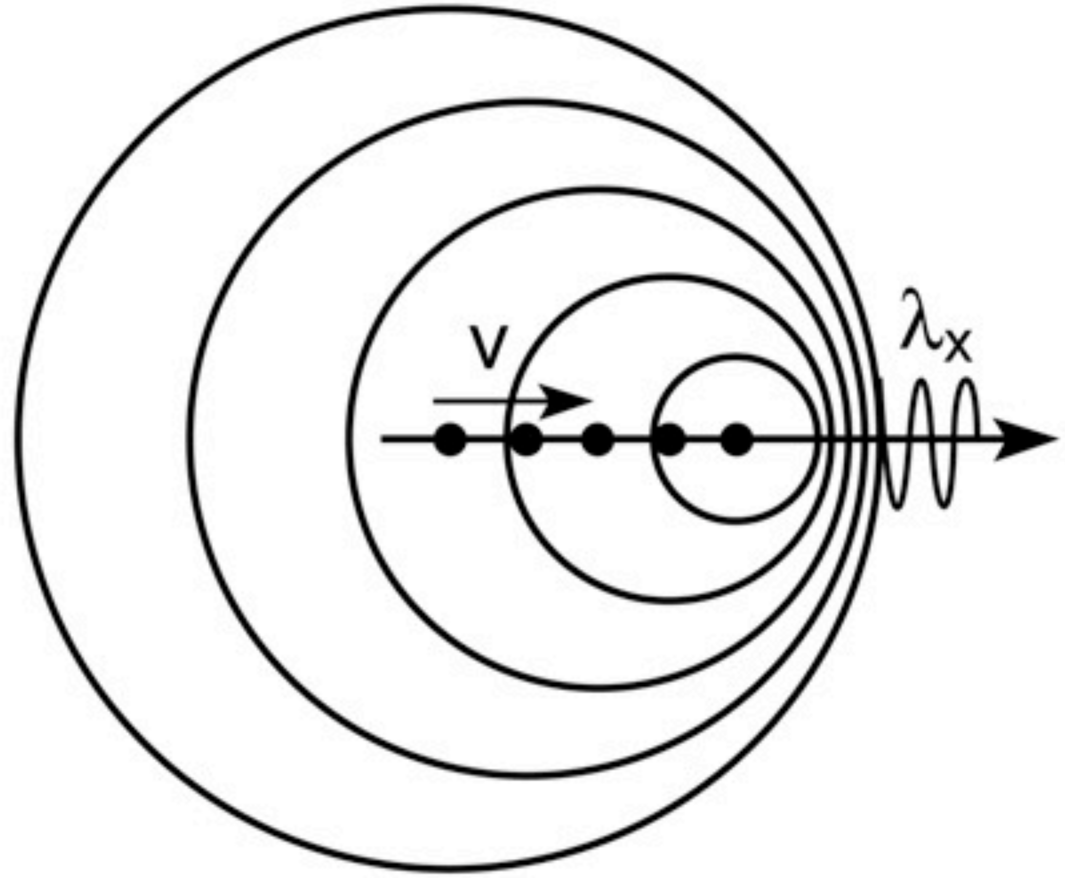

$$\lambda = \lambda' \left(1 - \frac{v}{c}\cos\theta\right)$$

# Doppler shift

v<<c




v≲c



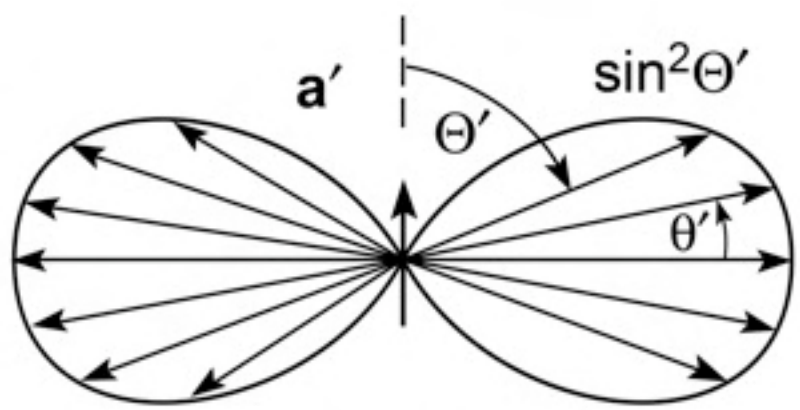

$$\lambda = \lambda' \left(1 - \frac{v}{c} \cos\theta\right)$$

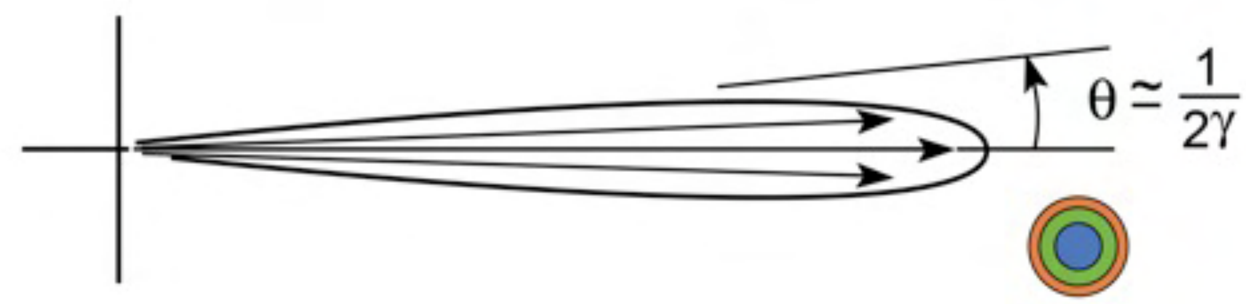
$$\lambda = \lambda' \gamma \left(1 - \frac{v}{c} \cos\theta\right)$$

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1 - \beta^2}} =$$

$$= \frac{E}{m_0 c^2} \approx 1957 E\,(GeV)$$

# Angle transformation




$$\tan\theta = \frac{\sin\theta'}{\gamma\left(\beta + \cos\theta'\right)}$$

$$\theta \approx \frac{1}{2\gamma}$$

# Lorentz transformations

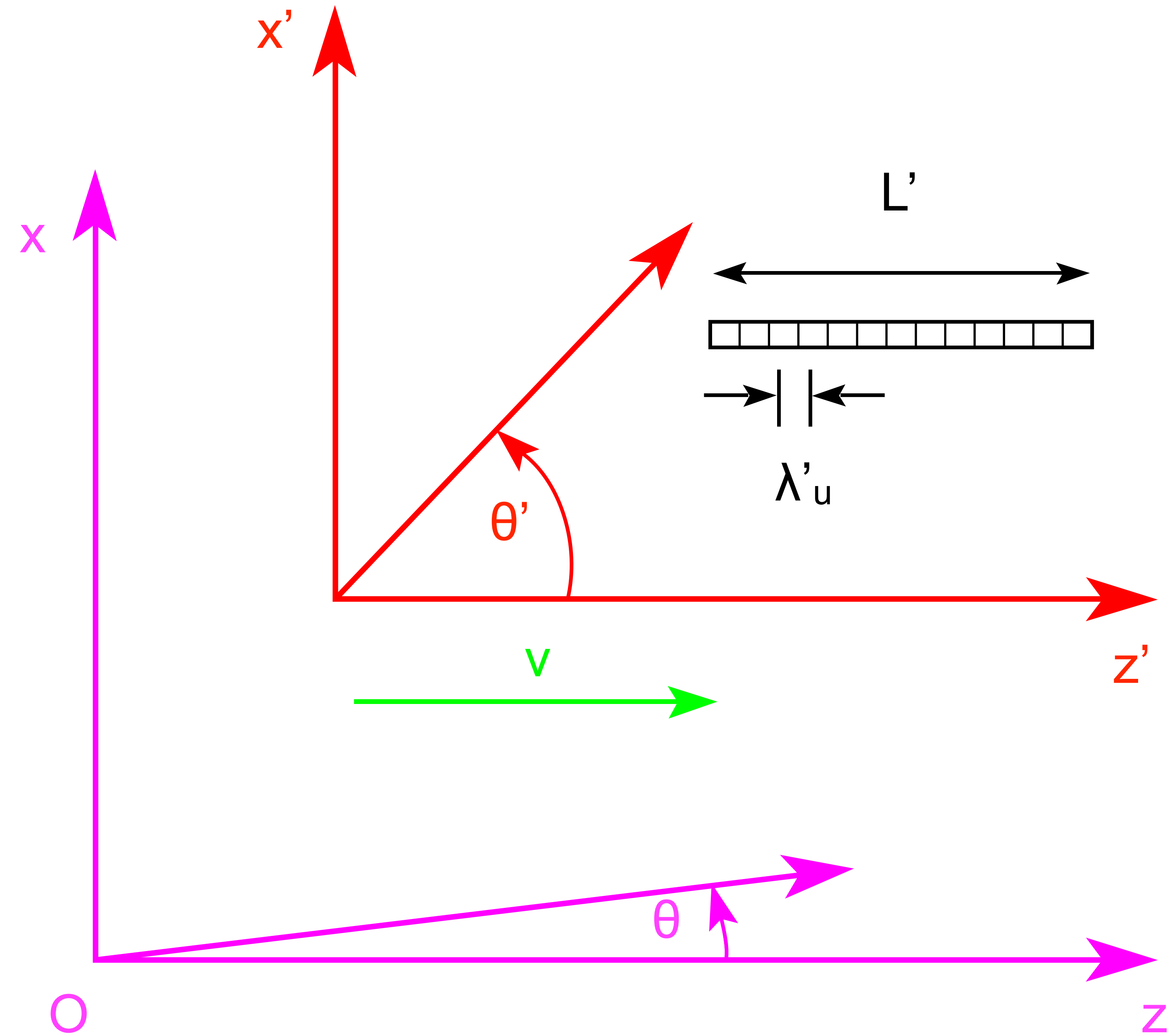
$$z = \gamma \left( z' + \beta c t' \right)$$

$$t = \gamma \left( t' + \frac{\beta z'}{c} \right)$$

$$y = y' \text{ and } x = x'$$

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$



# Doppler shift

$$e^{i\phi} = e^{i\left(\omega t - \vec{k}\cdot\vec{r}\right)}$$

$$\phi = \omega t - k_z z - k_x x - k_y y$$

$$\phi' = \omega' t' - k'_z z' - k'_x x' - k'_y y'$$

the two phases must be equal (e.g they could two wave crests)

$$\phi' = \phi$$

$$\omega = \gamma\left(\omega' + \beta c k'_z\right)$$

$$k_z = \gamma\left(k'_z + \frac{\beta}{c}\omega'\right)$$

$$\omega = \omega'\gamma\left(1 + \beta\cos\theta'\right)$$

$$k_y = k'_y \text{ and } k_x = k'_x$$

# Angular transformations

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

$$\sin\theta = \frac{\sin\theta'}{\gamma\left(1 + \beta\cos\theta'\right)}$$

$$\tan\theta = \frac{\sin\theta'}{\gamma\left(\beta + \cos\theta'\right)}$$

# Useful formulas

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}} \; ; \quad \beta = \frac{v}{c}$$

$$E_e = \gamma m c^2, \quad \boldsymbol{p} = \gamma m \mathbf{v}$$

$$\gamma = \frac{E_e}{mc^2} = 1957\, E_e(\text{GeV})$$

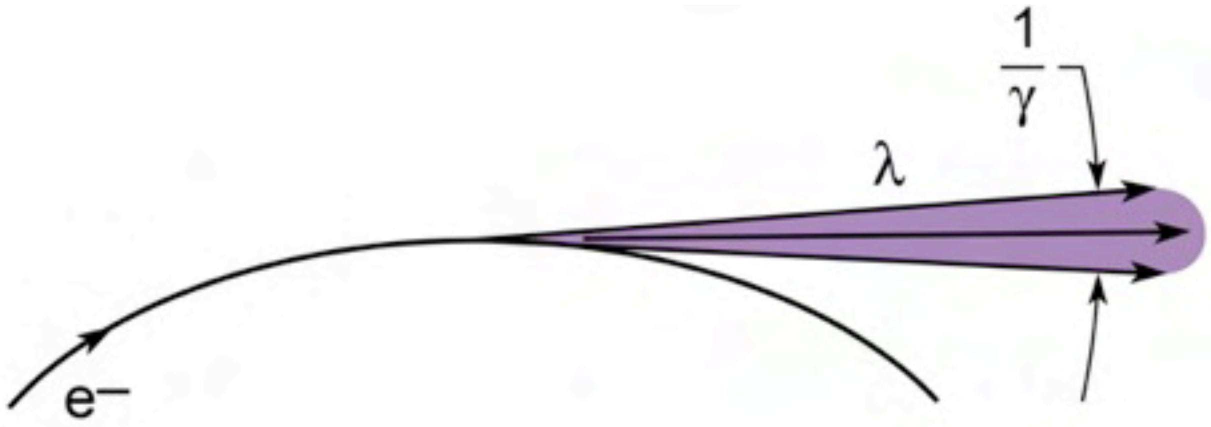
$$\hbar\omega \cdot \lambda = 1239.842 \text{ eV} \cdot \text{nm}$$

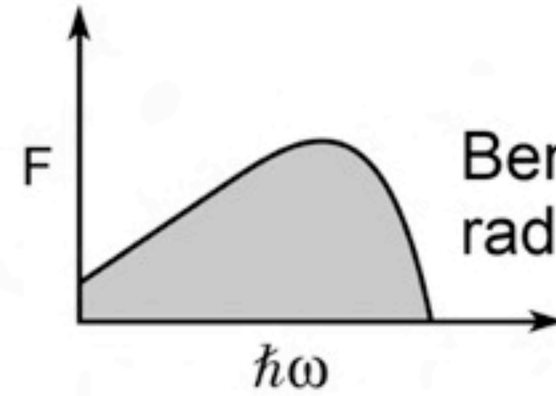
$$1 \text{ watt} \Rightarrow 5.034 \times 10^{15} \lambda[\text{nm}] \frac{\text{photons}}{\text{s}}$$

Bending Magnet: $E_c = \dfrac{3e\hbar B\gamma^2}{2m}$ , $E_c(\text{keV}) = 0.6650 E_e^2(\text{GeV}) B(\text{T})$

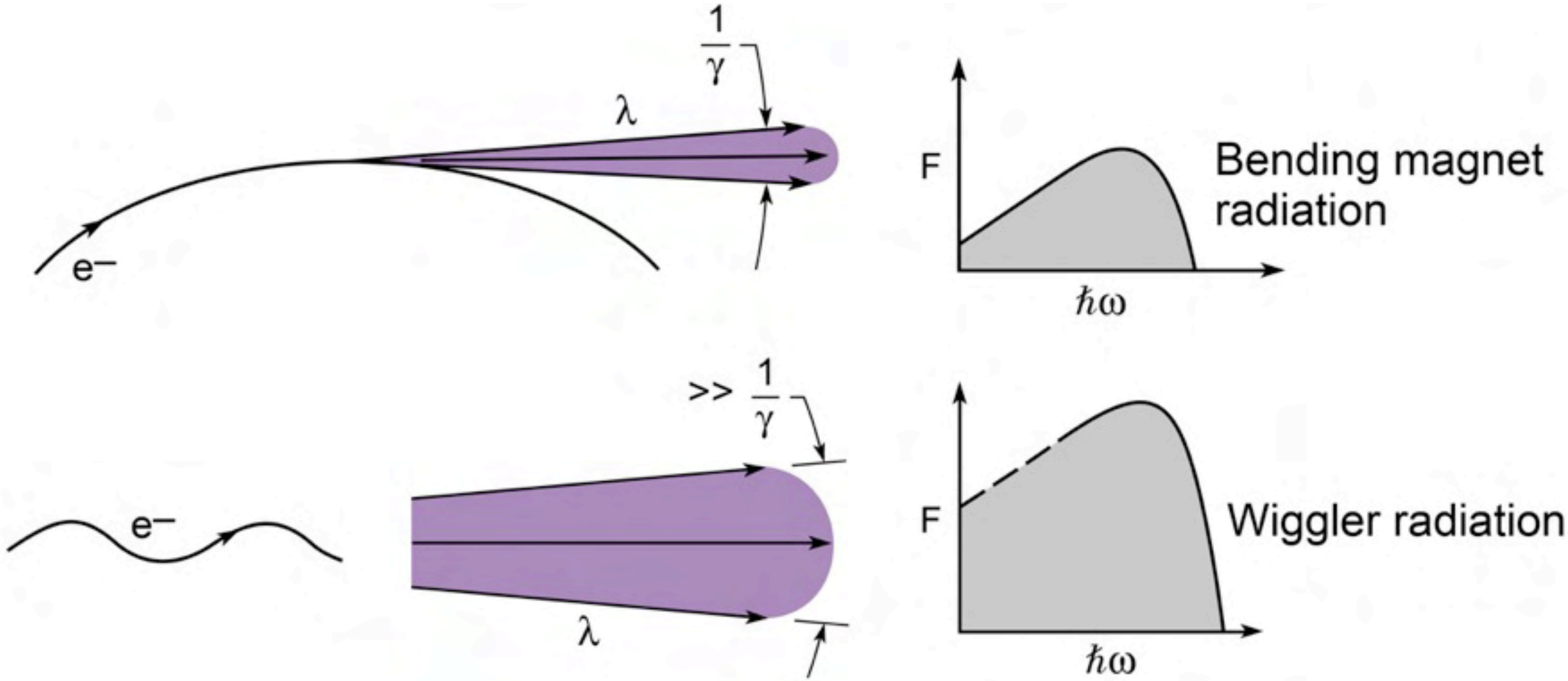
Undulator: $\lambda = \dfrac{\lambda_u}{2\gamma^2}\left(1 + \dfrac{K^2}{2} + \gamma^2\theta^2\right)$; $E(\text{keV}) = \dfrac{0.9496 E_e^2(\text{GeV})}{\lambda_u(\text{cm})\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)}$

where $K \equiv \dfrac{eB_0\lambda_u}{2\pi mc} = 0.9337 B_0(\text{T})\lambda_u(\text{cm})$

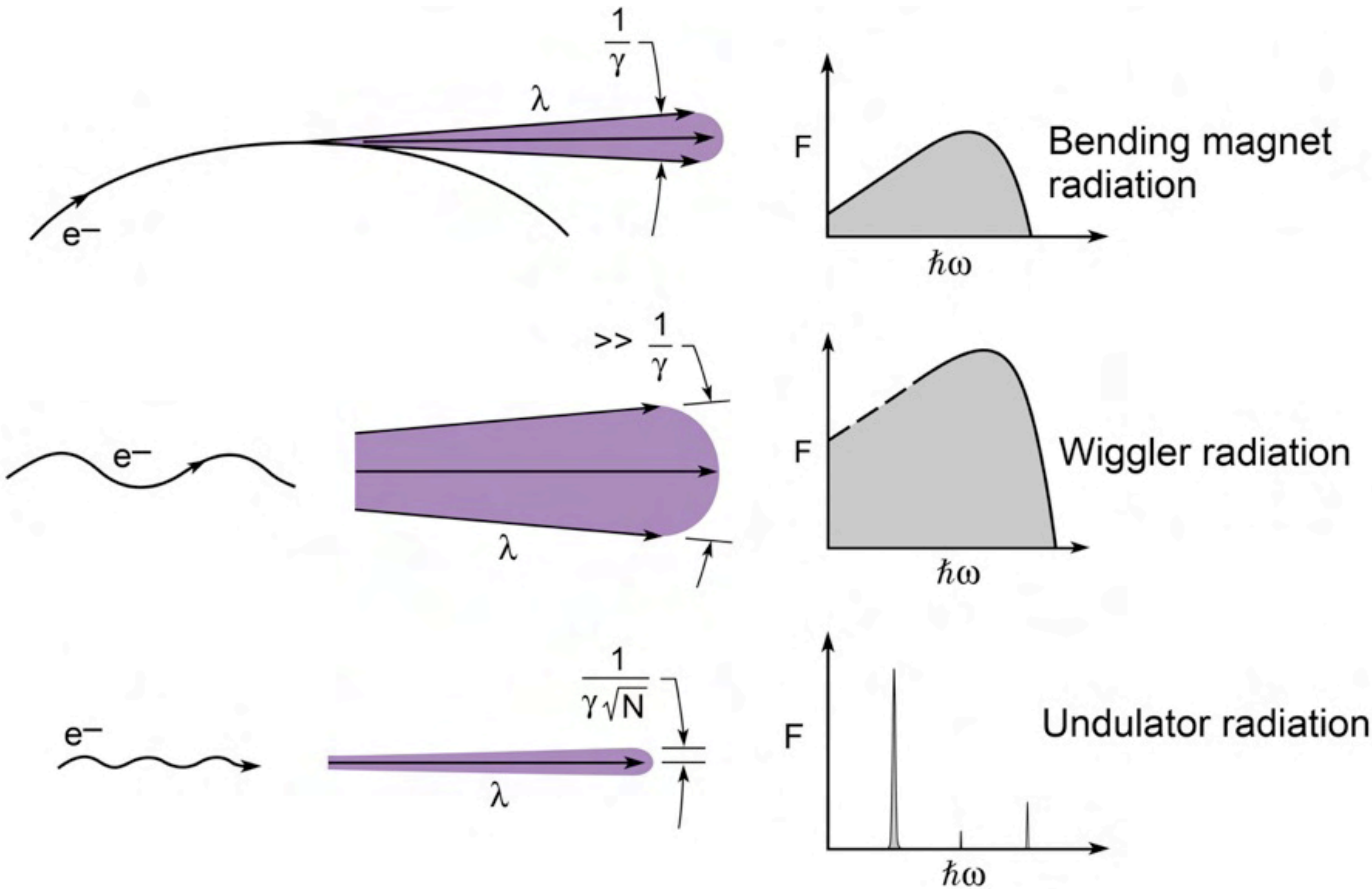



Bending magnet radiation



1/γ
λ
e−
F
ħω
Bending magnet radiation
>> 1/γ
e−
λ
F
ħω
Wiggler radiation


1/γ
λ
e−
F
ħω
Bending magnet radiation
>> 1/γ
e−
λ
F
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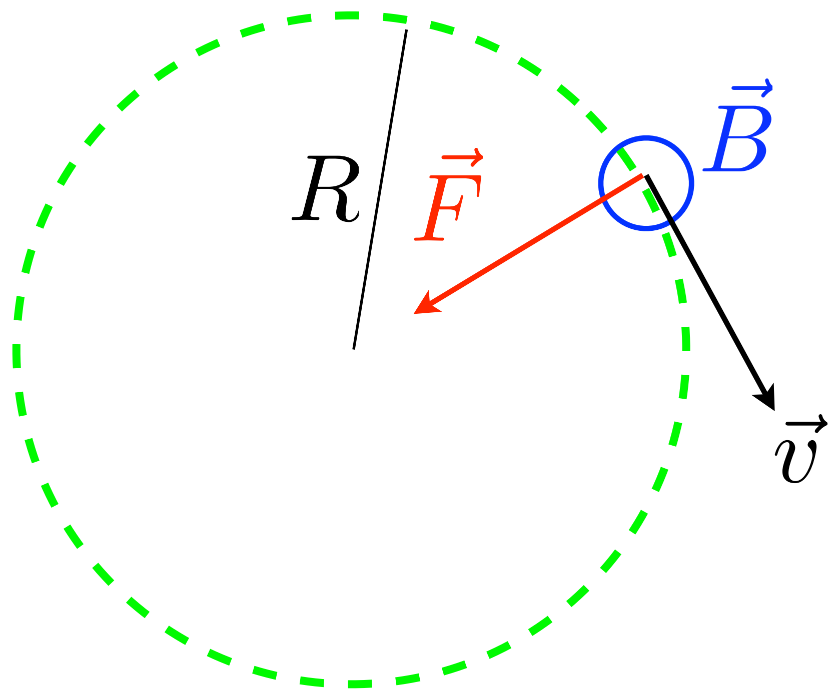
Wiggler radiation
1/(γ√N)
e−
λ
F
ħω
Undulator radiation

# An electron in a magnetic field

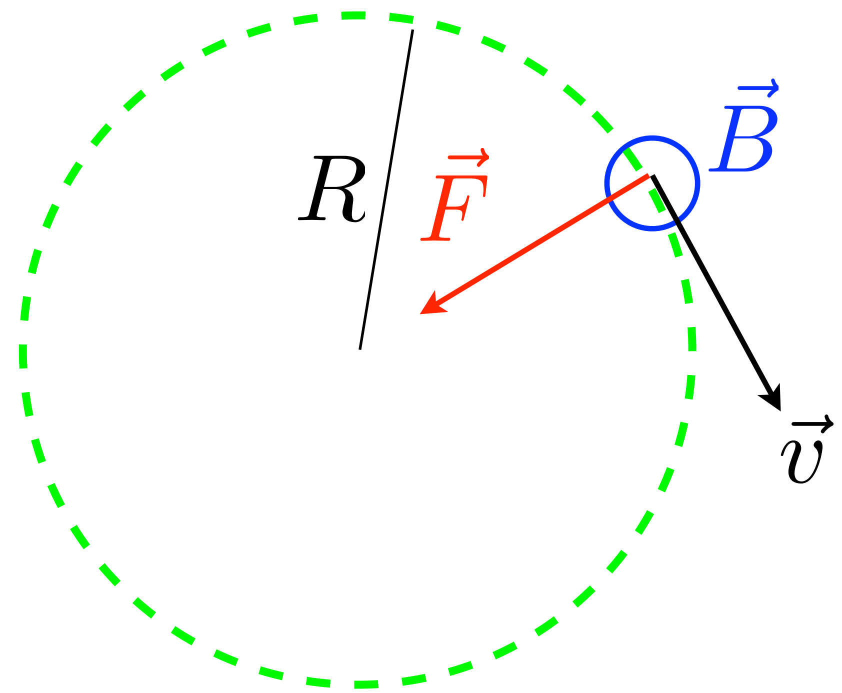
The force is given by:



# An electron in a magnetic field

The force is given by:

$$\vec{F} = \frac{d\vec{p}}{dt} = -e\vec{v} \times \vec{B}$$



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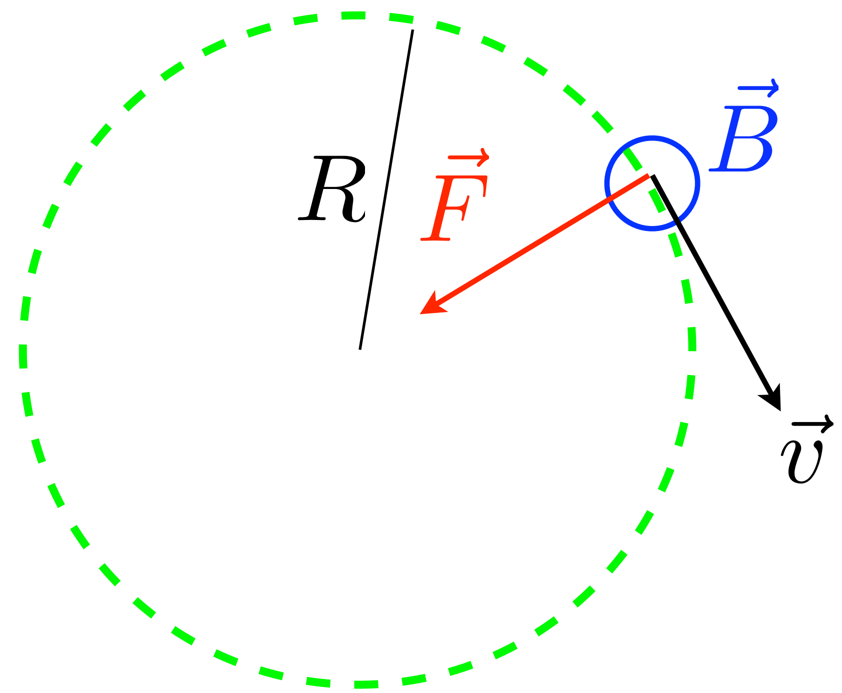
where the momentum is

$$\vec{p} = \gamma m \vec{v}$$



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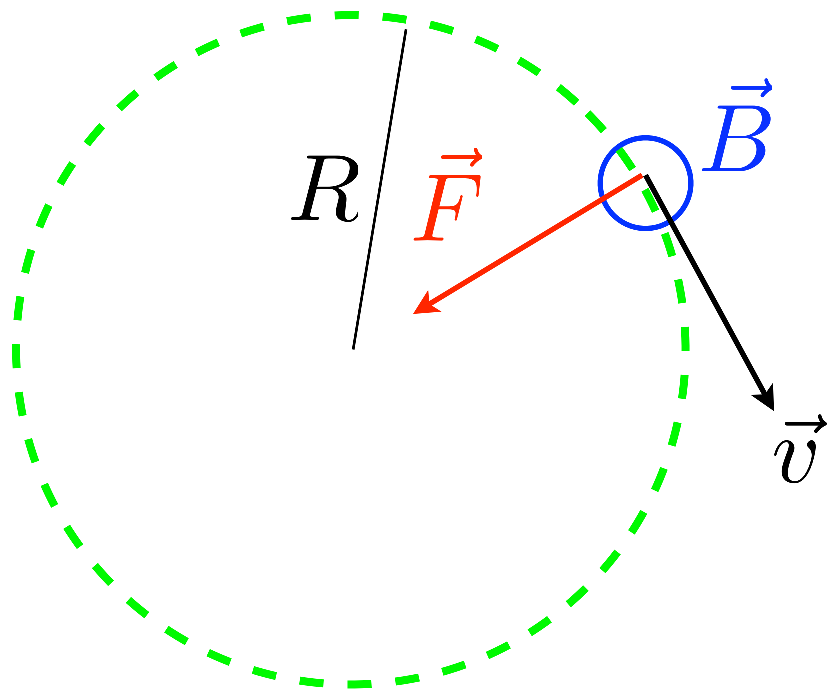


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a magnetic field does not change the energy so

$$\frac{d\vec{p}}{dt} = \gamma m\frac{d\vec{v}}{dt} = -e\vec{v} \times \vec{B}$$

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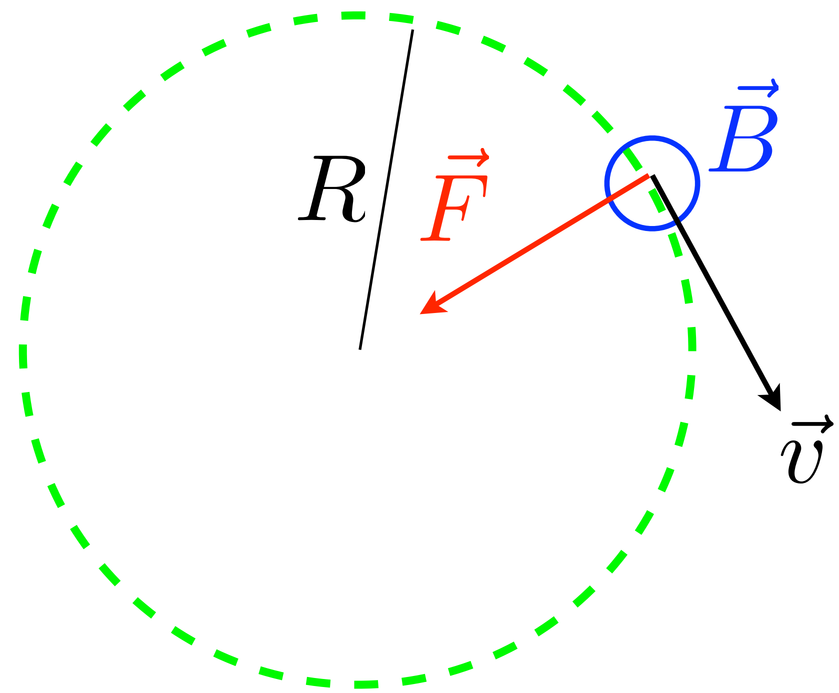


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$$\frac{d\vec{p}}{dt} = \gamma m \frac{d\vec{v}}{dt} = -e\vec{v} \times \vec{B}$$

therefore

$$\gamma m \left( -\frac{v^2}{R} \right) = -evB$$

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The force is given by:

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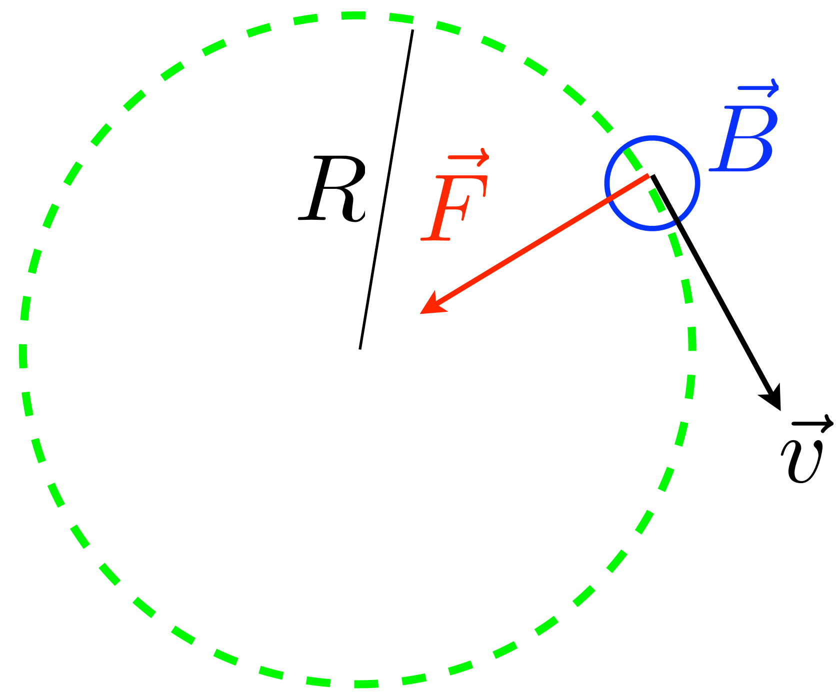


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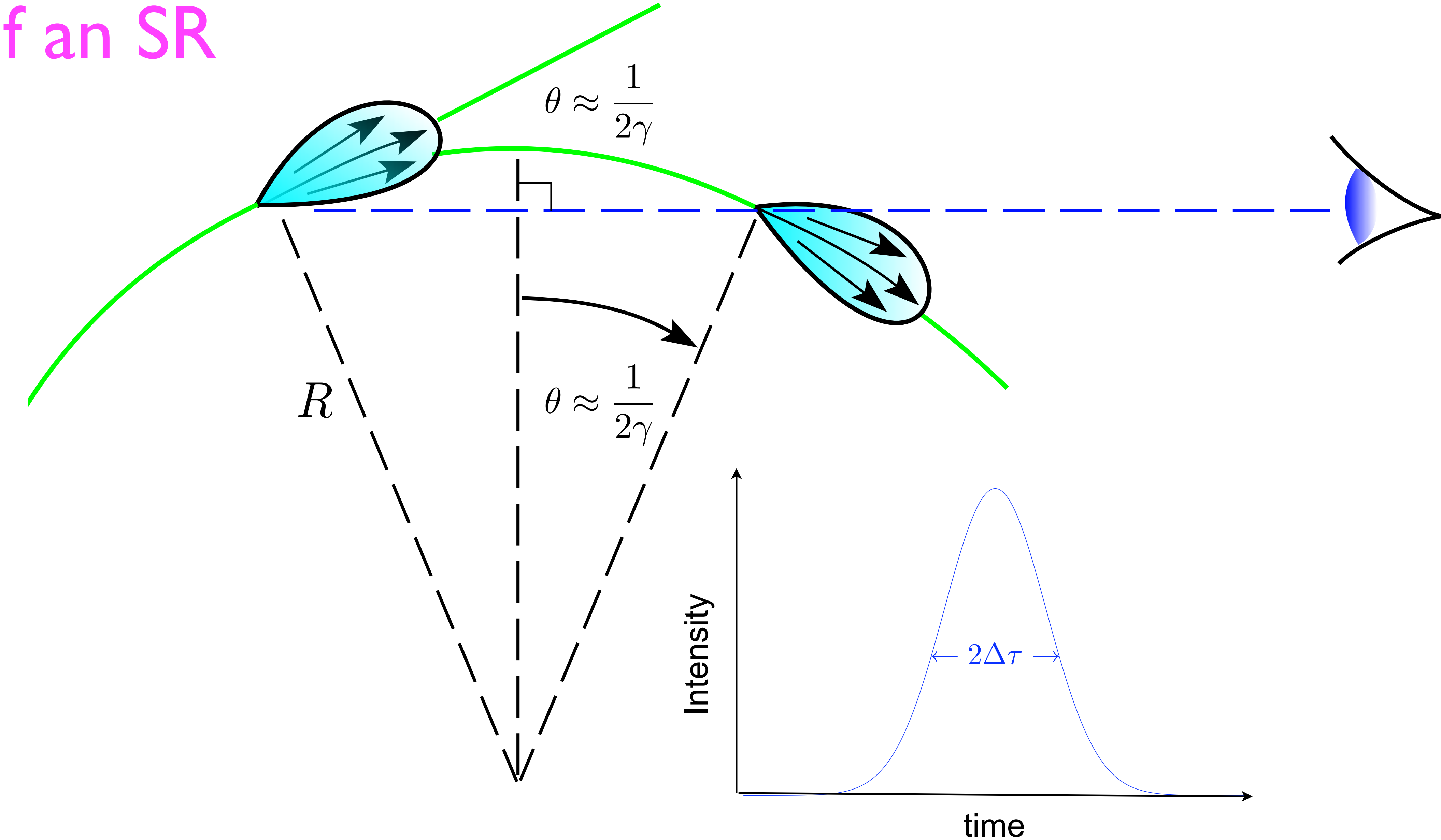
therefore

$$\gamma m \left( -\frac{v^2}{R} \right) = -evB$$

and: $$R = \frac{\gamma m v}{eB} \simeq \frac{\gamma m c}{eB}$$

# Production of an SR pulse



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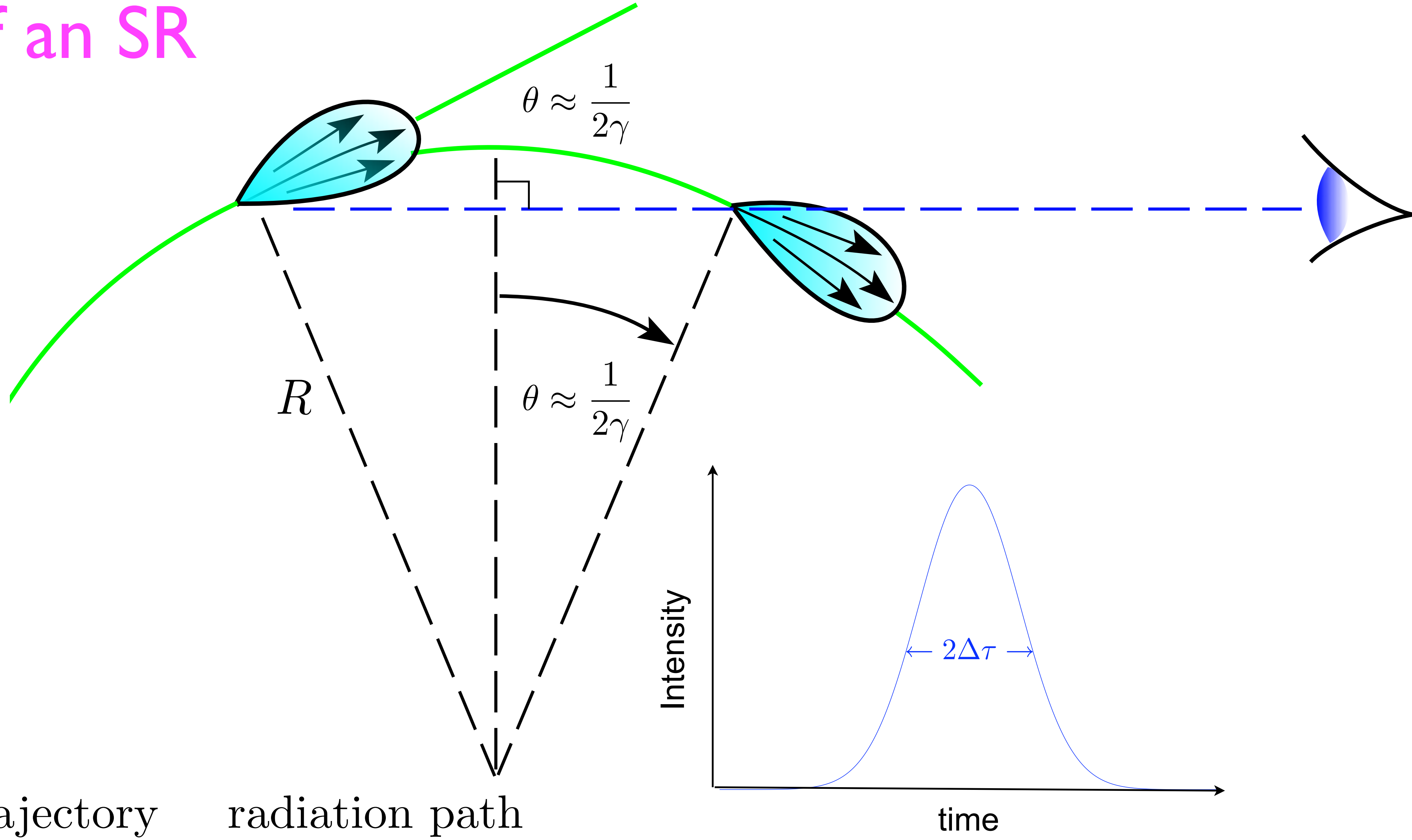
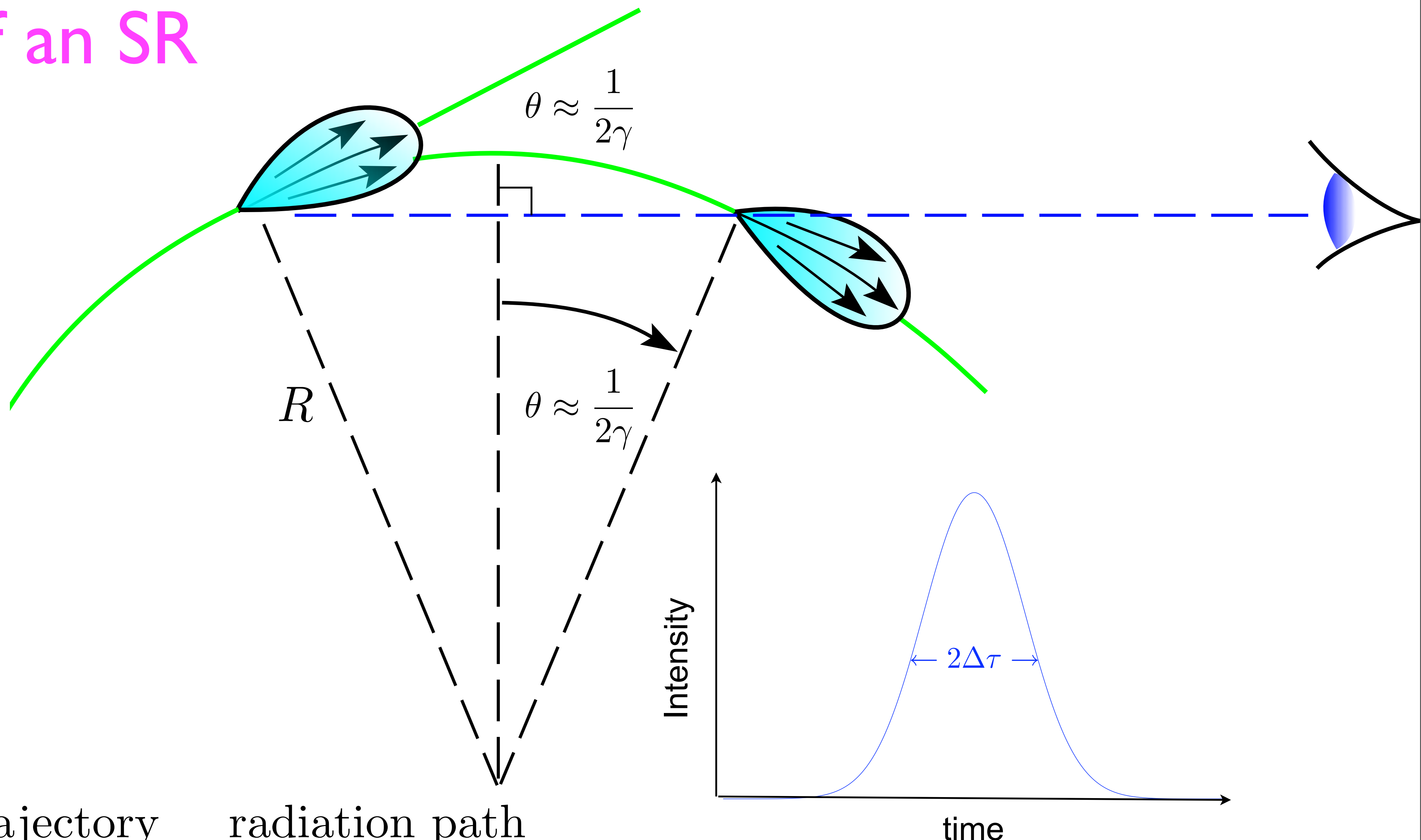



$$2\Delta\tau = \frac{\text{electron trajectory}}{v} - \frac{\text{radiation path}}{c}$$

# Production of an SR pulse




$$2\Delta\tau = \frac{\text{electron trajectory}}{v} - \frac{\text{radiation path}}{c}$$

$$2\Delta\tau \simeq \frac{R \cdot 2\theta}{v} - \frac{2R\sin\theta}{c}$$

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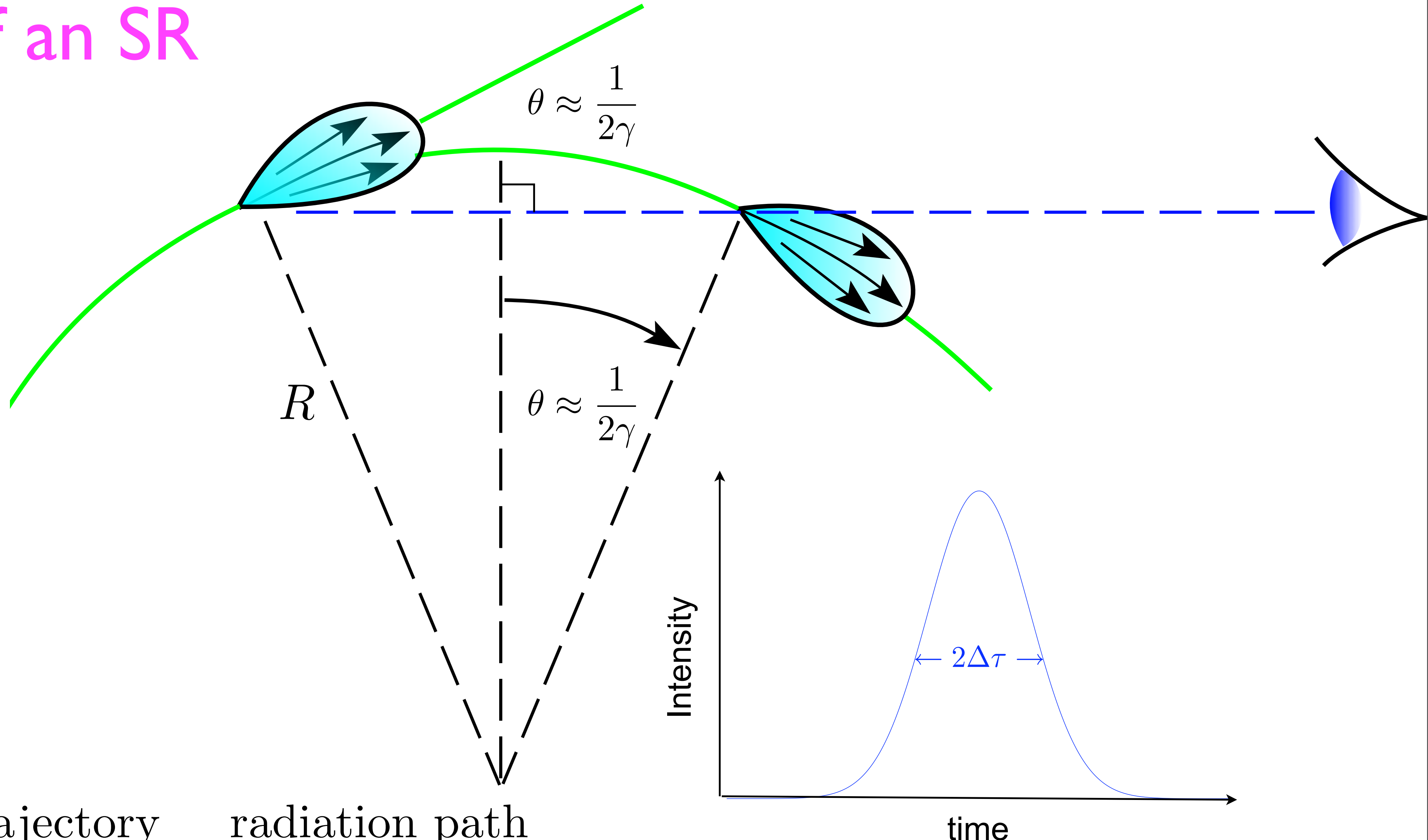


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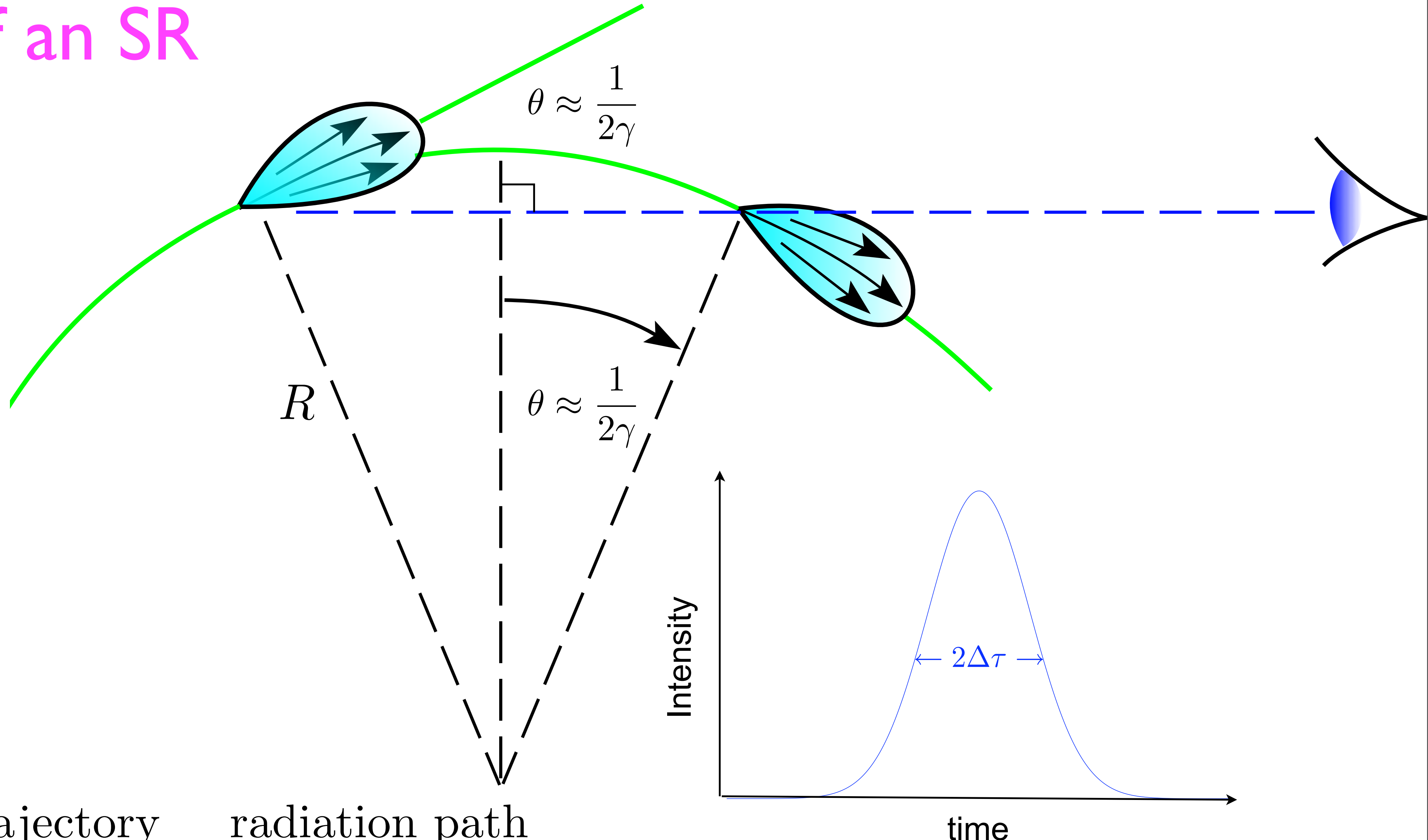
$$\theta \simeq 1/2\gamma \qquad \sin\theta \simeq \theta$$

# Production of an SR pulse

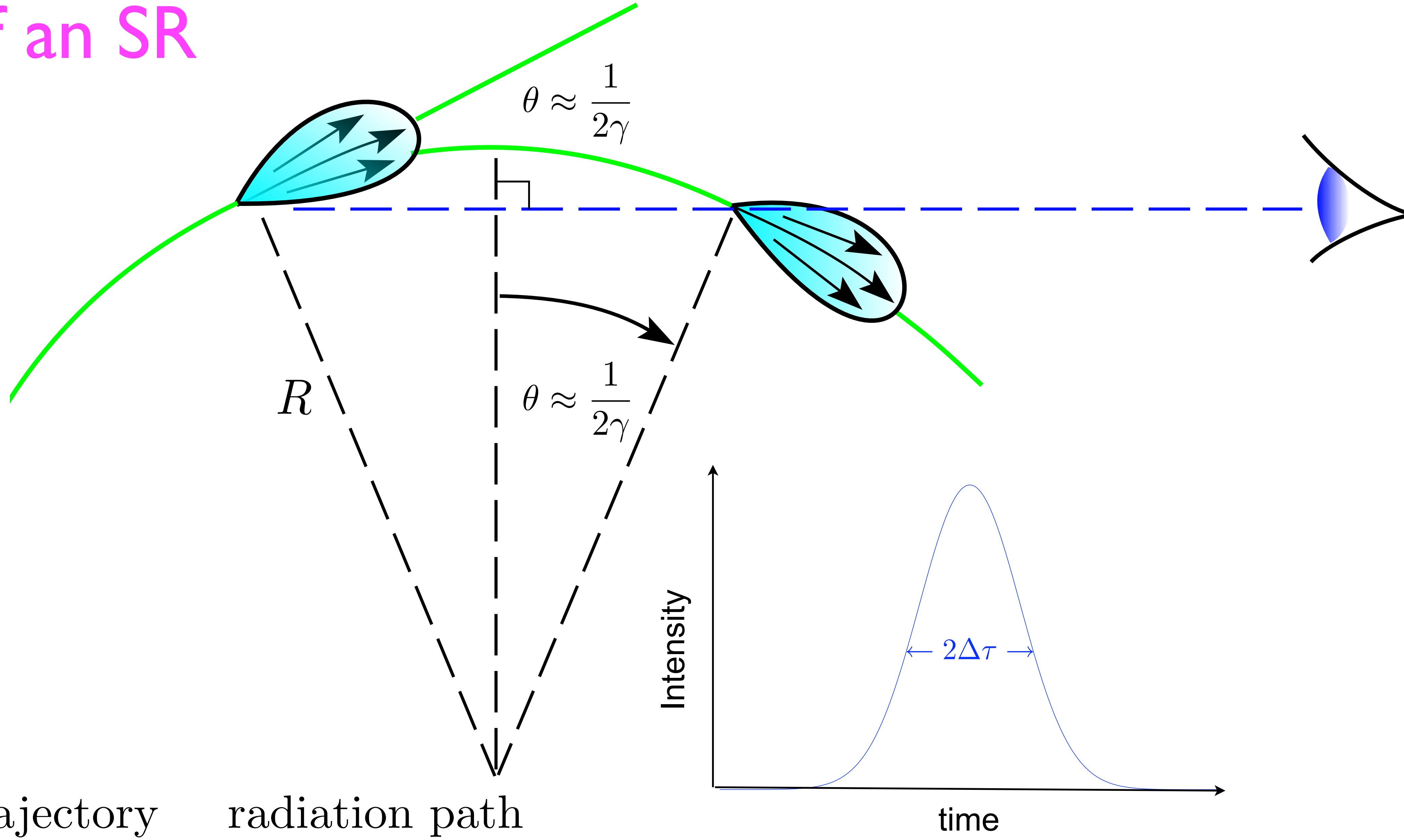



$$2\Delta\tau = \frac{\text{electron trajectory}}{v} - \frac{\text{radiation path}}{c}$$

$$2\Delta\tau \simeq \frac{R \cdot 2\theta}{v} - \frac{2R\sin\theta}{c} \qquad \theta \simeq 1/2\gamma \qquad \sin\theta \simeq \theta$$

$$2\Delta\tau \simeq \frac{R}{\gamma}\left(\frac{1}{v} - \frac{1}{c}\right) = \frac{R}{\gamma\beta c}(1-\beta)$$

# Production of an SR pulse

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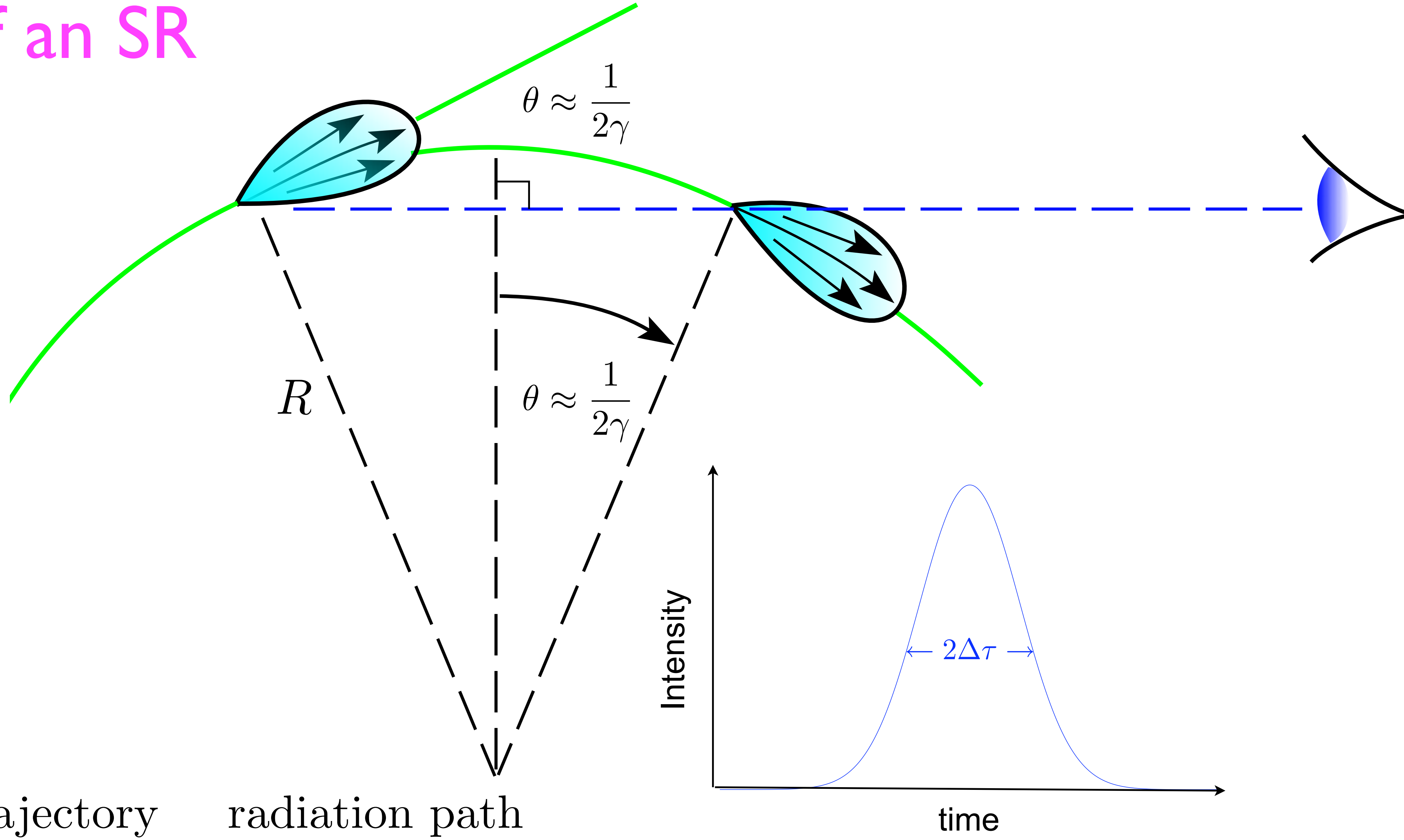


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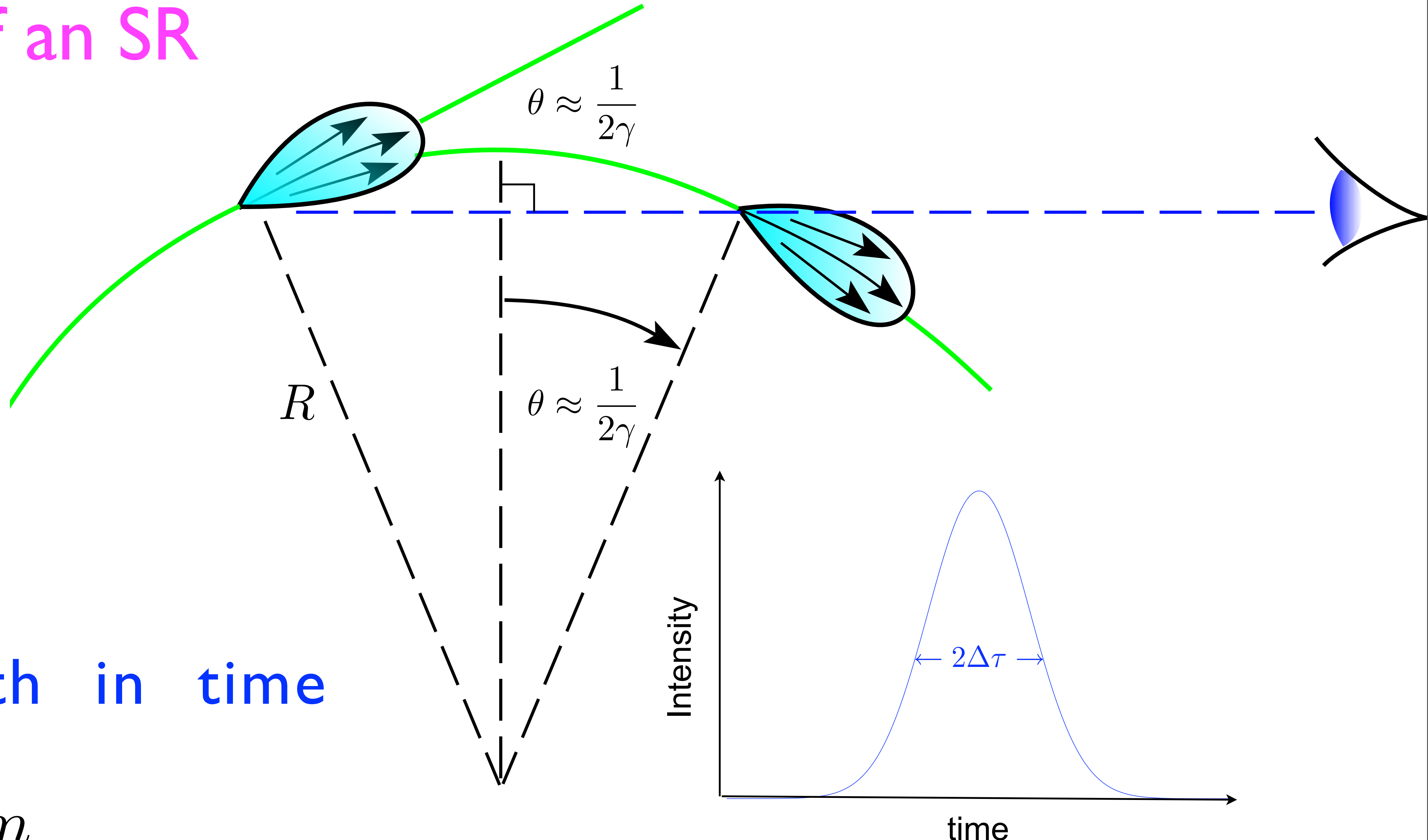
$$2\Delta\tau \approx \frac{m}{2eB\gamma^2}$$

# Production of an SR pulse



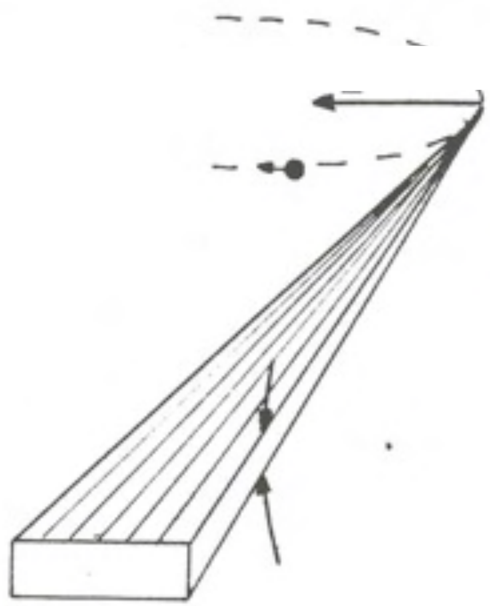

A pulse width in time domain

$$2\Delta\tau \approx \frac{m}{2eB\gamma^2}$$

Corresponds to a photon energy distribution over a range

$$\Delta E \approx \frac{\hbar}{2\Delta\tau} = \frac{2e\hbar B\gamma^2}{m}$$

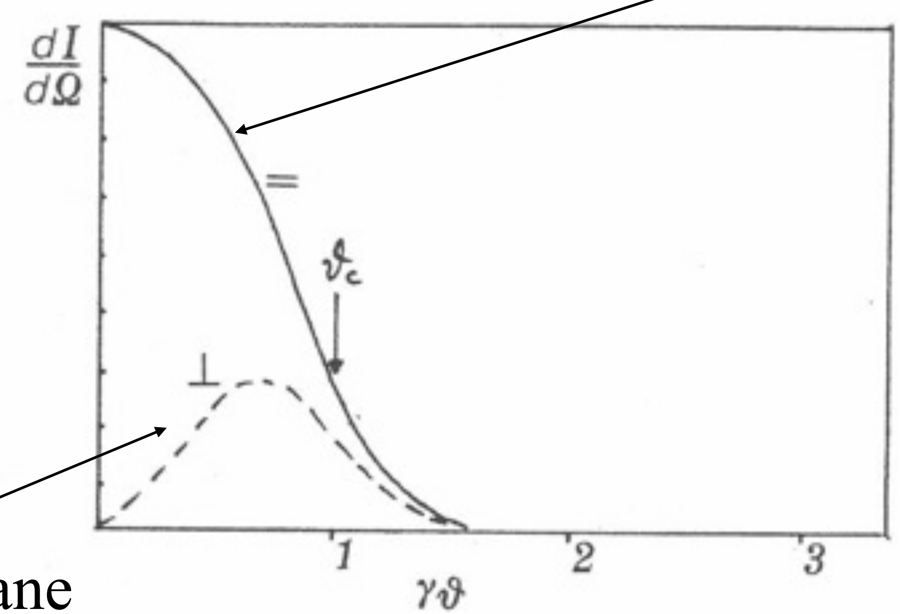
# Synchrotron radiation emitted by a bending magnet



High directionality

Light polarized in the (x,z) plane

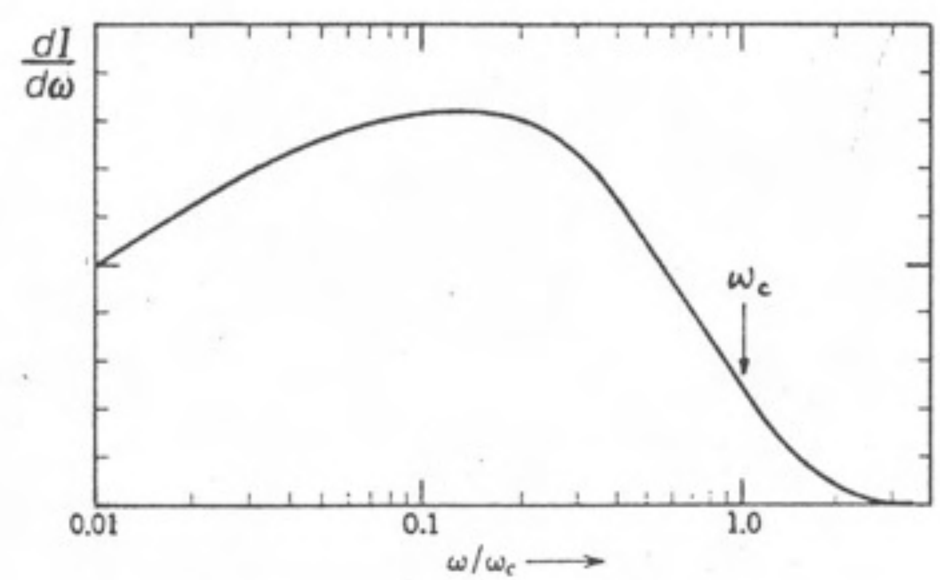
$\frac{dn}{d\Omega}$




Defined polarization

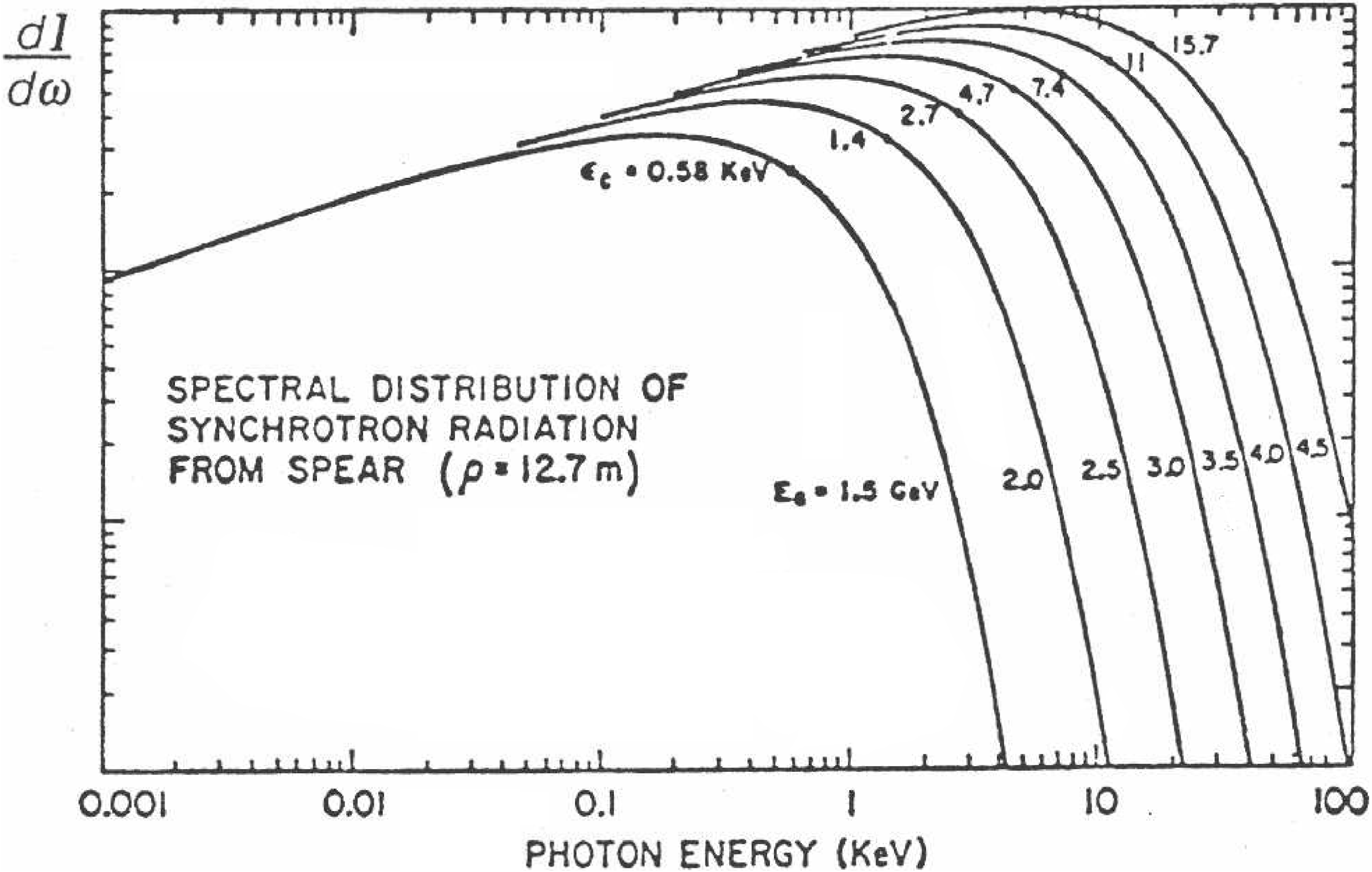
Light polarized in the (y,z) plane

Broad band

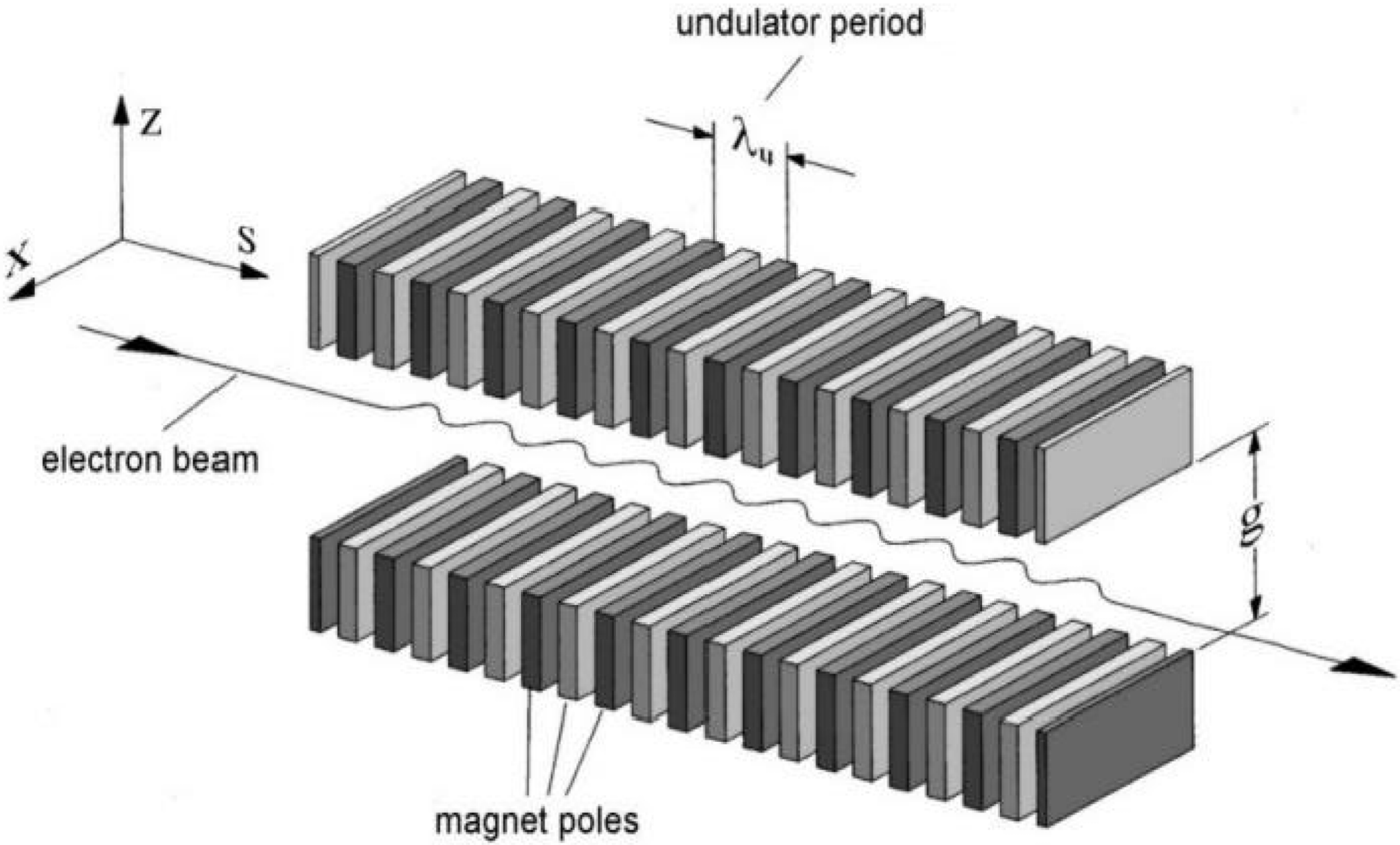


# Bending magnet radiation:

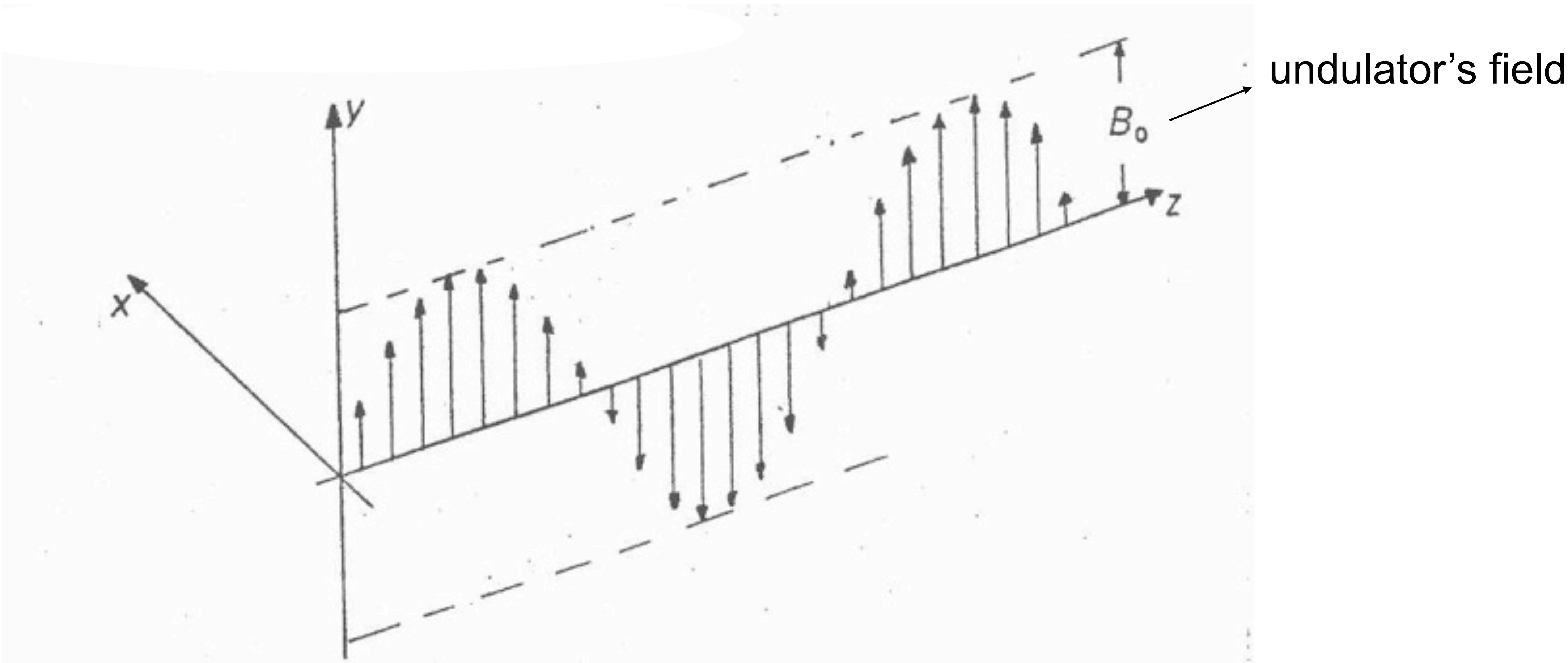
## Spectral distribution for different beam energies

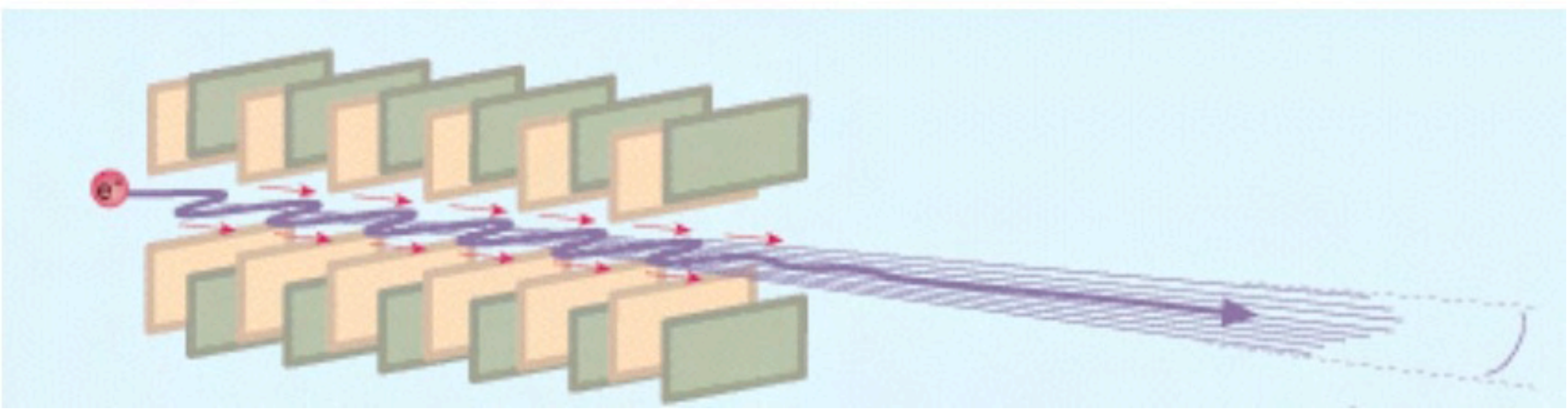


# Undulators & Wigglers



# Undulators



# Undulators & Wigglers

$$K = \frac{\lambda_u e B_0}{2\pi m_0 c}$$

$$\Theta_{\max} = \frac{K}{\gamma}$$

$$K < 1 \Rightarrow \Theta_{\max} < 1/\gamma$$

$$K > 1 \Rightarrow \Theta_{\max} > 1/\gamma$$

# Spectral profile

The radiation emitted on axis ( $\vartheta = 0$) by the particle is characterized by

$$\lambda_i = \frac{\lambda_0}{2i\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

$\lambda_0$ undulator's period

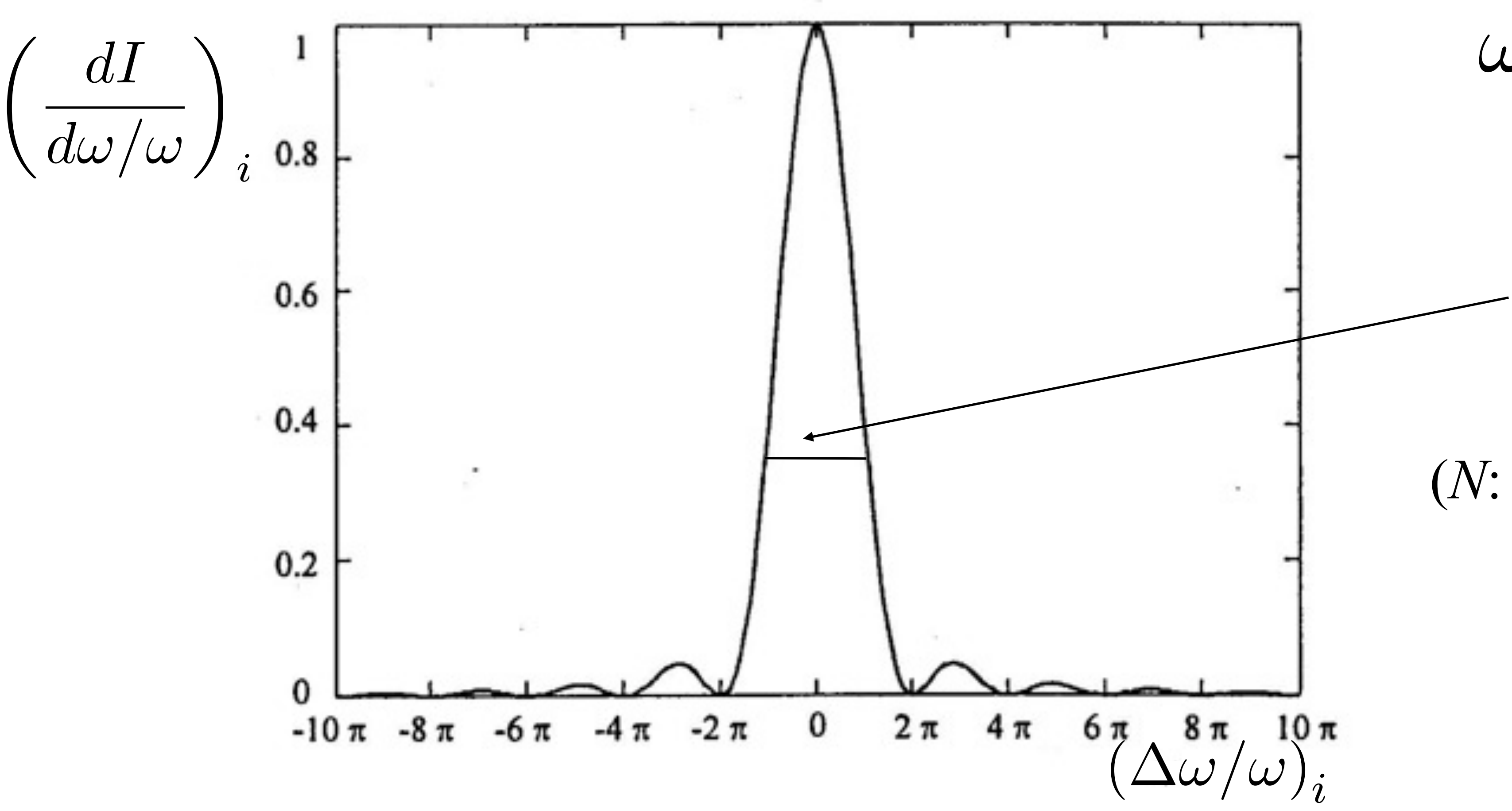
$i$ harmonic number

$\gamma$ electrons' energy

$K \propto \lambda_0 B_0$ undulator's strength

$B_0$ undulator's field

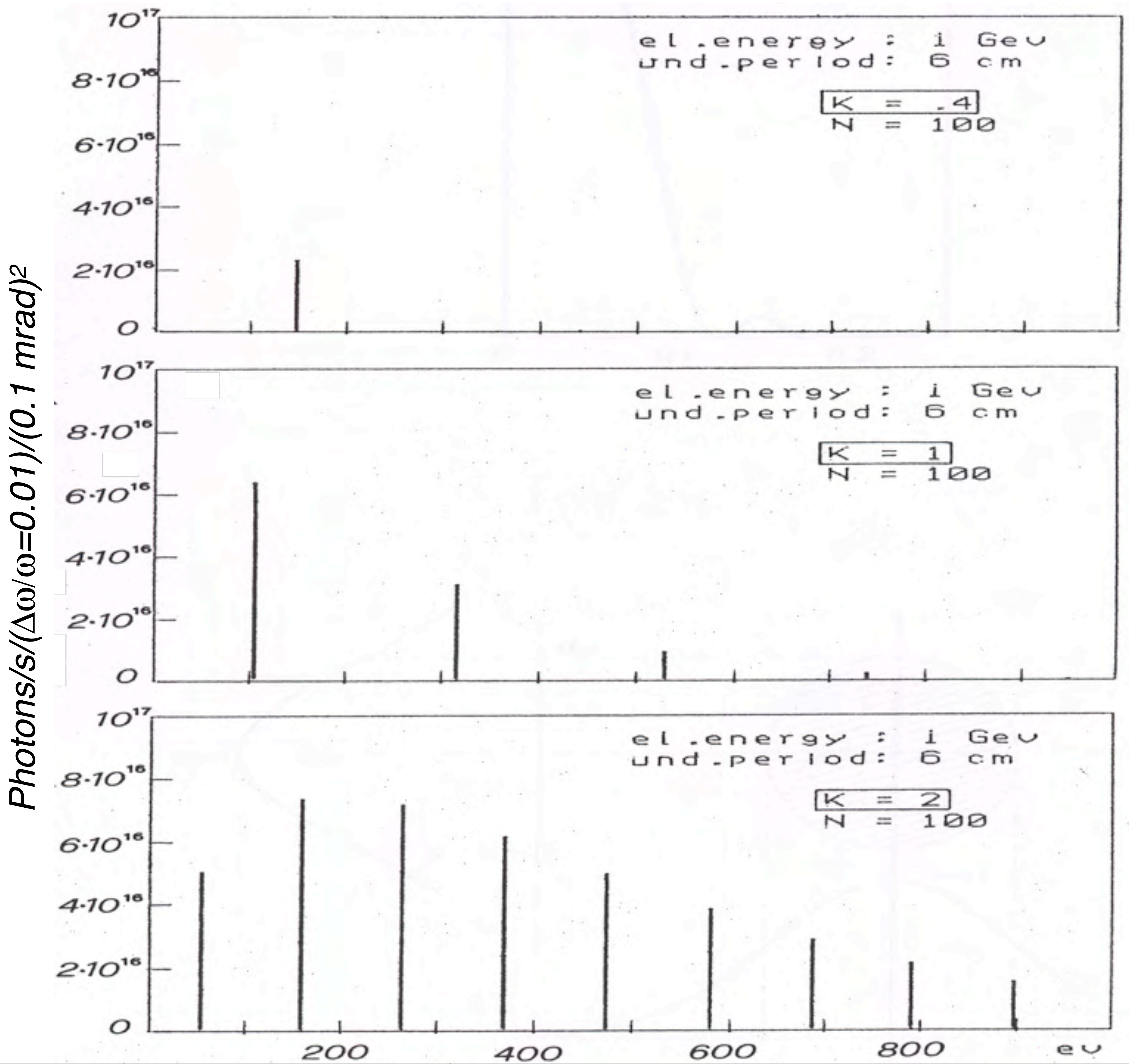
$$\omega_i = \frac{2\pi c}{\lambda_i}$$




$$\left(\frac{\sigma_\omega}{\omega}\right)_i \approx \frac{1}{iN}$$

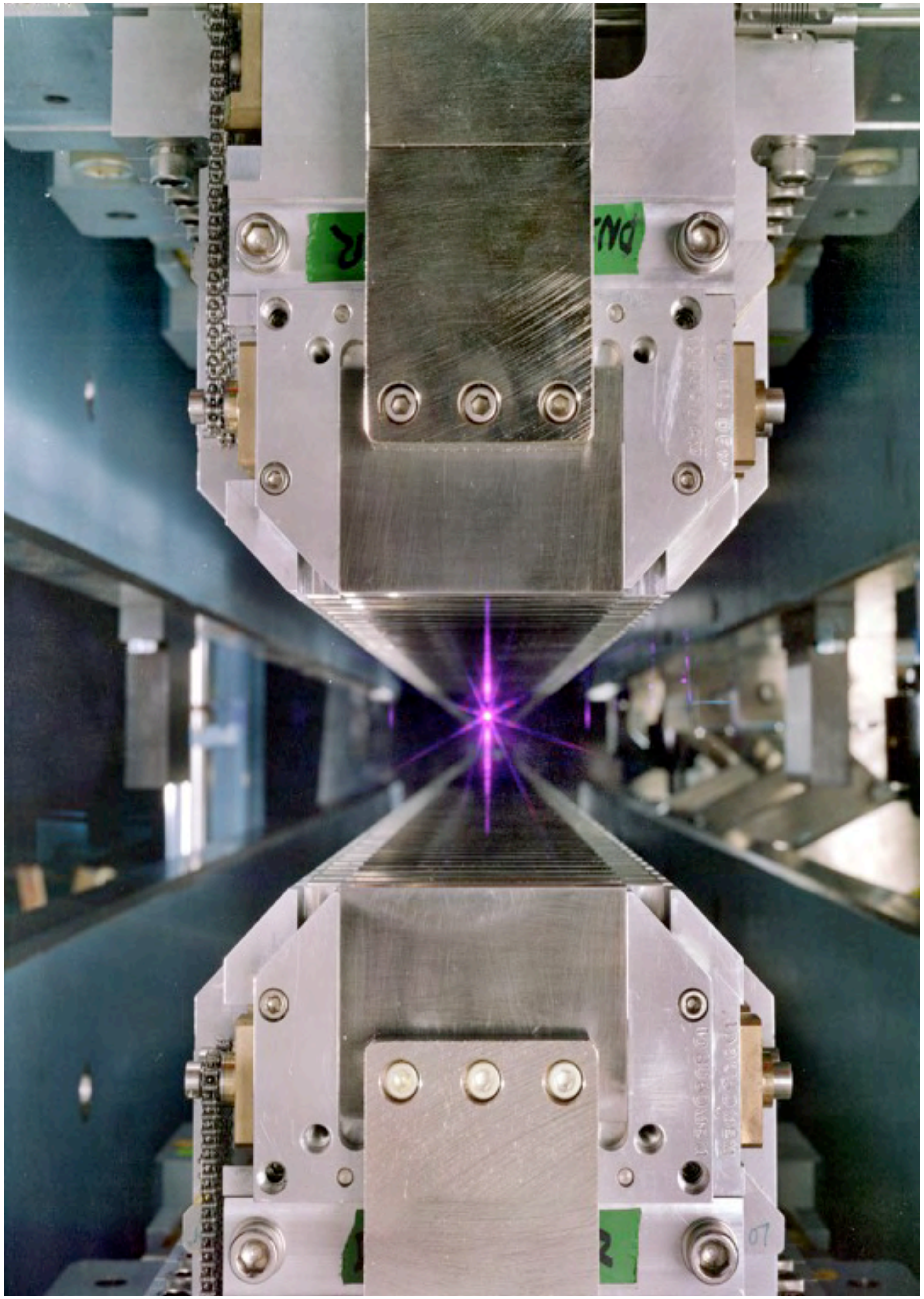
($N$: number of undulator's periods)

# Spectral profile for different K values



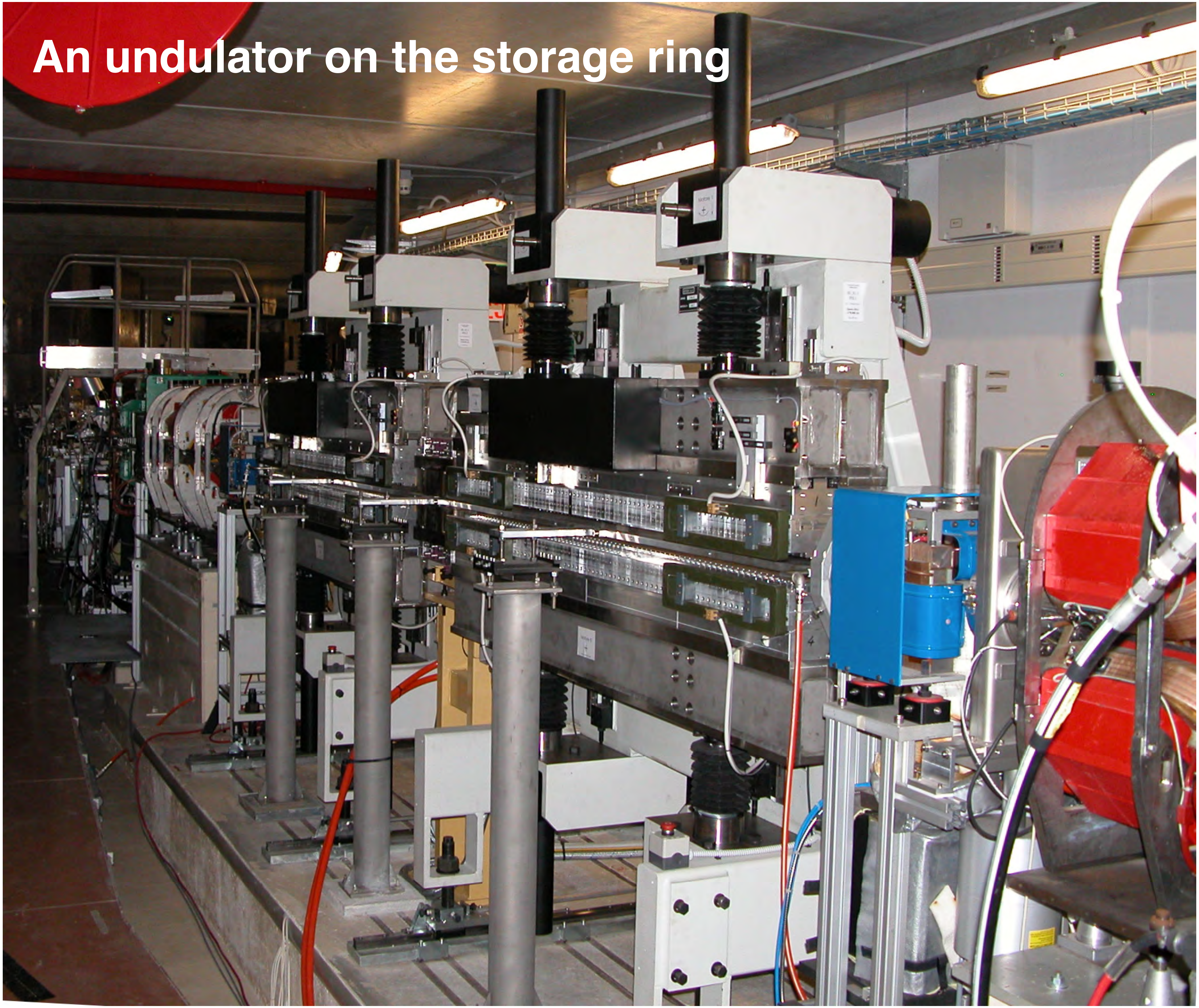
# 3rd generation synchrotron radiation sources

| Source | Energy (GeV ) | Emittance (nm rad) | Circumference (m) |
|---|---|---|---|
| MAX II | 1.5 | 9 | 90 |
| ALS | 1.9 | 5.6 | 196.8 |
| BESSY II | 1.9 | 6.4 | 240 |
| ELETTRA | 2 | 7 | 258 |
| Swiss LS | 2.4 | 5 | 288 |
| NSLS | 2.5 | 50 | 170 |
| SOLEIL | 2.75 | 3.72 | 354 |
| Canadian LS | 2.9 | 18.2 | 170.4 |
| Australian LS | 3 | 6.88 | 216 |
| DIAMOND | 3 | 2.74 | 561.6 |
| ESRF | 6 | 4 | 844 |
| APS | 7 | 8.2 | 1104 |
| Spring-8 | 8 | 6 | 1436 |



# An undulator

# An undulator on the storage ring



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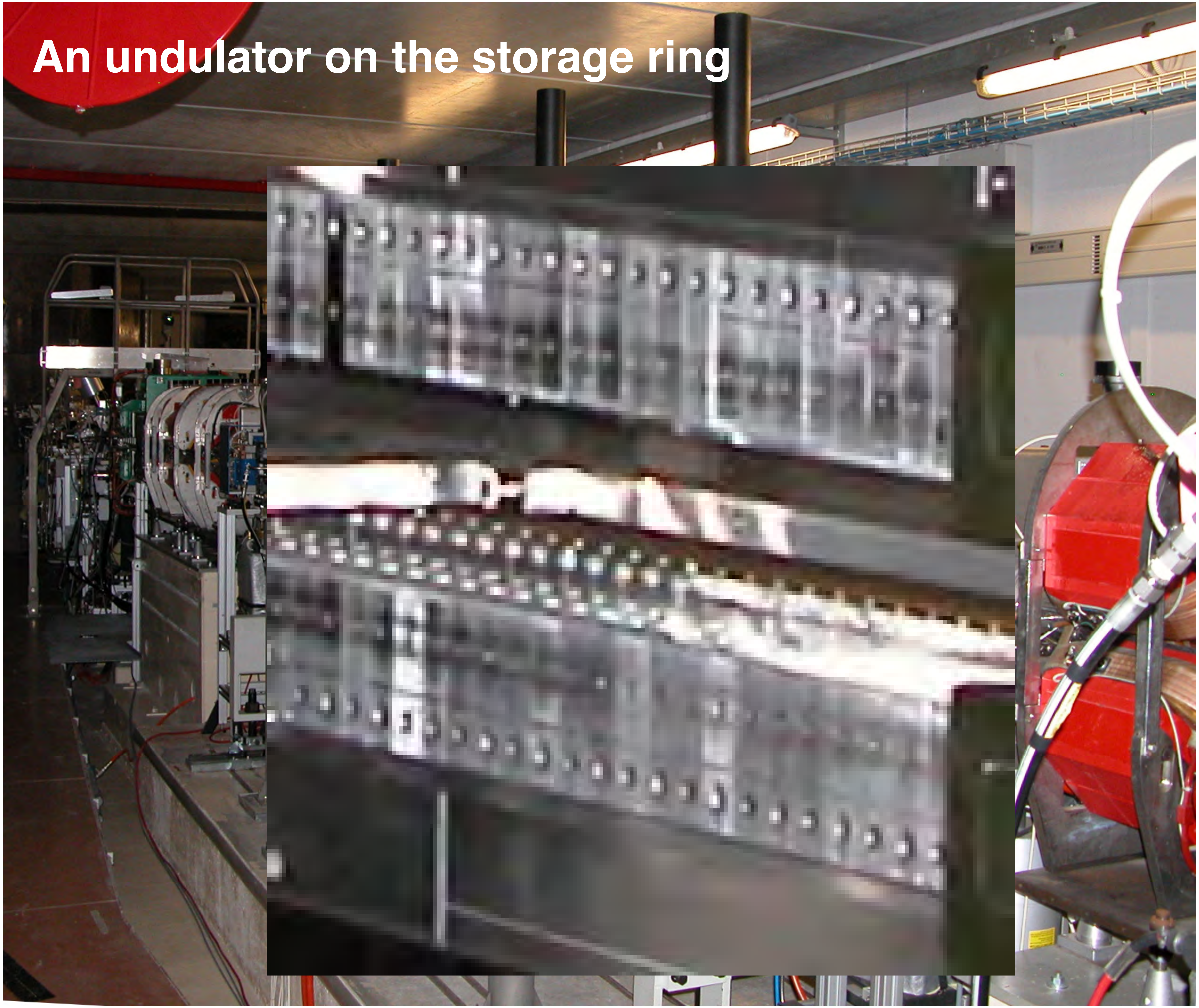

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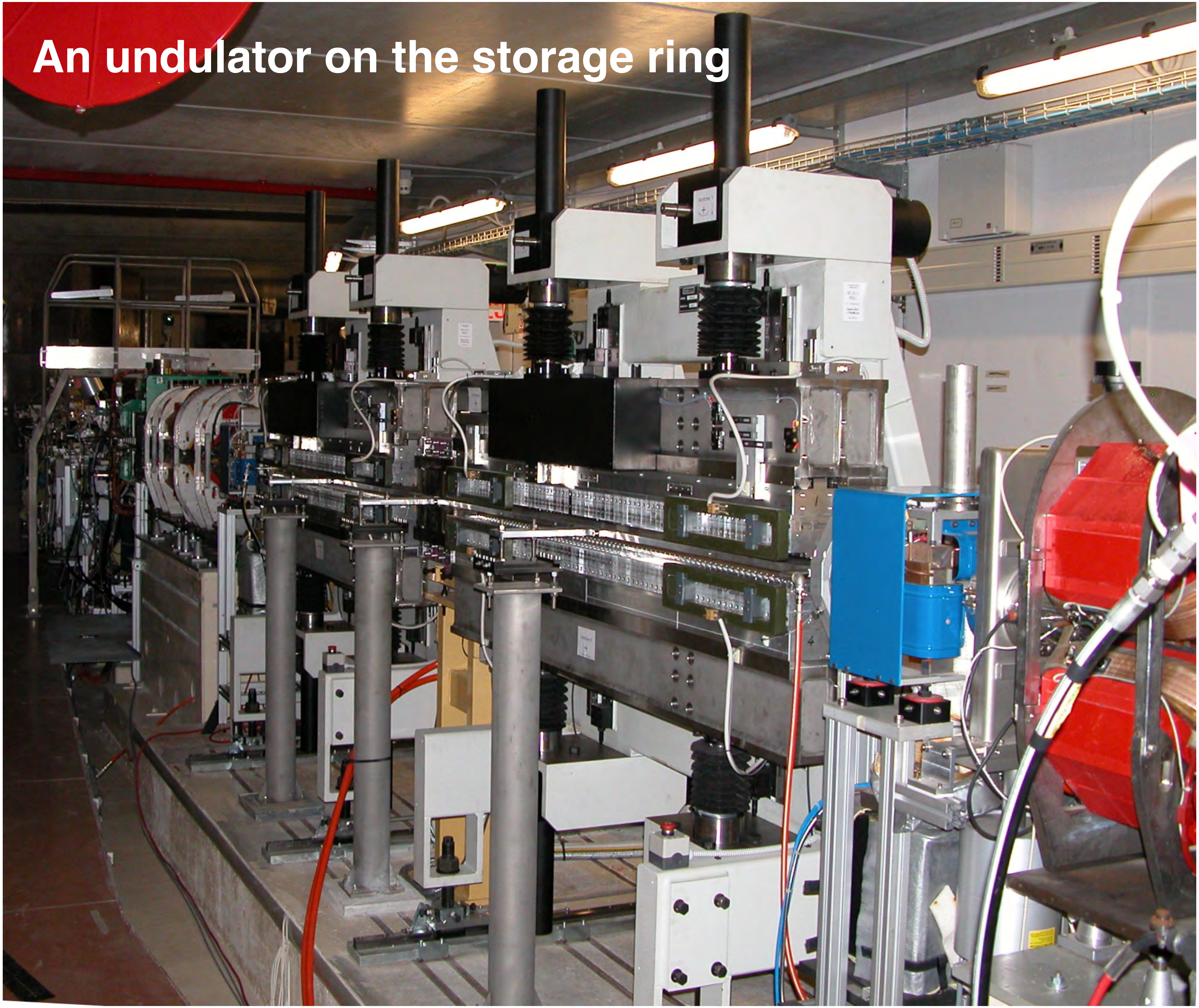
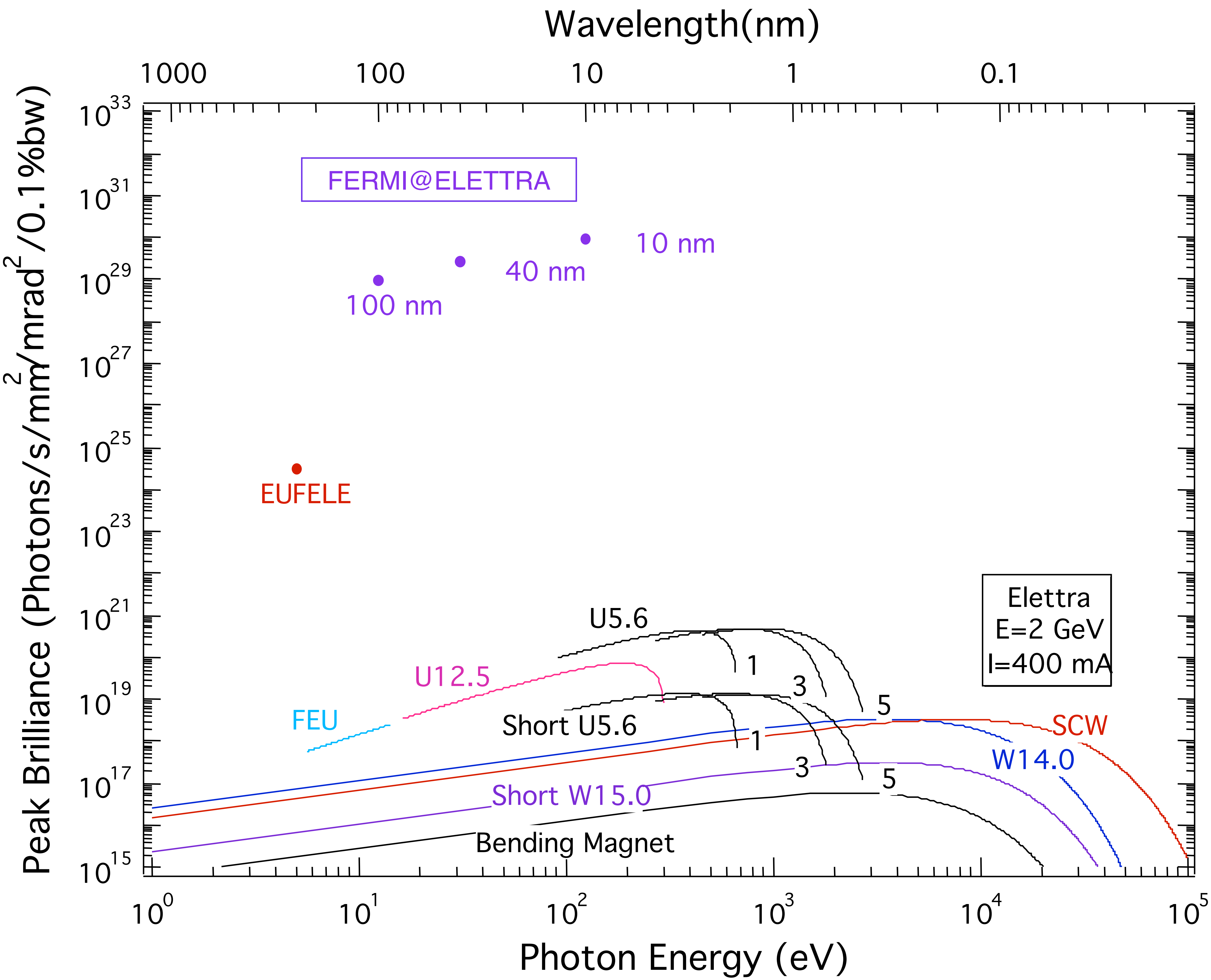


# Photon sources at elettra



# Contents

Lienard-Wiechert potentials

Angular distribution of power radiated by accelerated particles

- non-relativistic motion: Larmor's formula
- relativistic motion
- velocity || acceleration: bremsstrahlung
- velocity $\perp$ acceleration: synchrotron radiation

Angular and frequency distribution of energy radiated:

- the radiation integral
- radiation integral for bending magnet radiation
- radiation integral for undulator and wiggler radiation

Synchrotron light sources

- energy loss per turn
- characteristics of synchrotron radiation

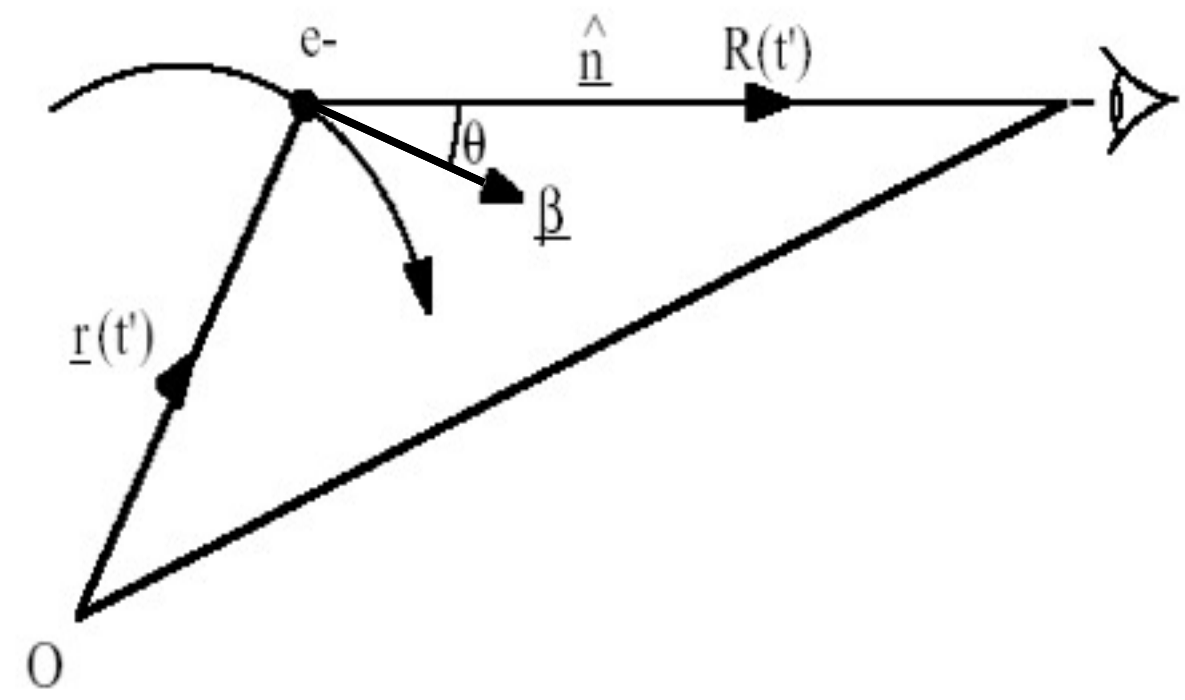
# Lienard-Wiechert Potentials

For a particle in motion the scalar and vector potentials take the Lienard -Wiechert form

$$\Phi(\bar{x},t) = \left[ \frac{e}{(1-\bar{\beta}\cdot\bar{n})R} \right]_{ret} \qquad \bar{A}(\bar{x},t) = \left[ \frac{e\bar{\beta}}{(1-\bar{\beta}\cdot\bar{n})R} \right]_{ret}$$

$[\ ]_{ret}$ means computed at “retarded time” t’

$$t = t' + \frac{R(t')}{c}$$



# Lineard-Wiechert Potentials (II)

The electric and magnetic fields are computed from the potentials

$$\bar{E} = -\nabla\Phi - \frac{\partial \bar{A}}{\partial t} \qquad \bar{B} = -\nabla \times \bar{A}$$

and are called Lineard-Wiechert fields

$$\bar{E}(\bar{x},t) = e\left[\frac{\bar{n}-\bar{\beta}}{\gamma^2(1-\bar{\beta}\cdot\bar{n})^3R^2}\right]_{rit} + e\left[\frac{\bar{n}\times(\bar{n}-\bar{\beta})\times\dot{\bar{\beta}}}{(1-\bar{\beta}\cdot\bar{n})^3R}\right]_{rit} \qquad \bar{B}(\bar{x},t) = \left[\,\bar{n}\times\bar{E}\,\right]_{rit}$$

Power radiated by a particle on a surface is the flux of the Poynting vector

$$\bar{S} = \frac{c}{4\pi}\bar{E}\times\bar{B} \qquad \Phi_{\Sigma}(\bar{S})(t) = \iint_{\Sigma}\bar{S}(\bar{x},t)\cdot\bar{n}d\Sigma$$

Angular distribution of radiated power

$$\frac{dP}{d\Omega} = (\bar{S}\cdot n)(1-\bar{n}\cdot\bar{\beta})R^2$$

radiation emitted by the particle

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velocity field

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velocity field

acceleration field

$$\propto \frac{1}{R} \qquad \vec{E} \perp \bar{B} \perp \hat{n}$$

Power radiated by a particle on a surface is the flux of the Poynting vector

$$\bar{S} = \frac{c}{4\pi}\bar{E}\times\bar{B} \qquad \Phi_\Sigma(\bar{S})(t) = \iint_\Sigma \bar{S}(\bar{x},t)\cdot\bar{n}\,d\Sigma$$

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$$\frac{dP}{d\Omega} = (\bar{S}\cdot n)(1 - \bar{n}\cdot\bar{\beta})R^2$$

radiation emitted by the particle

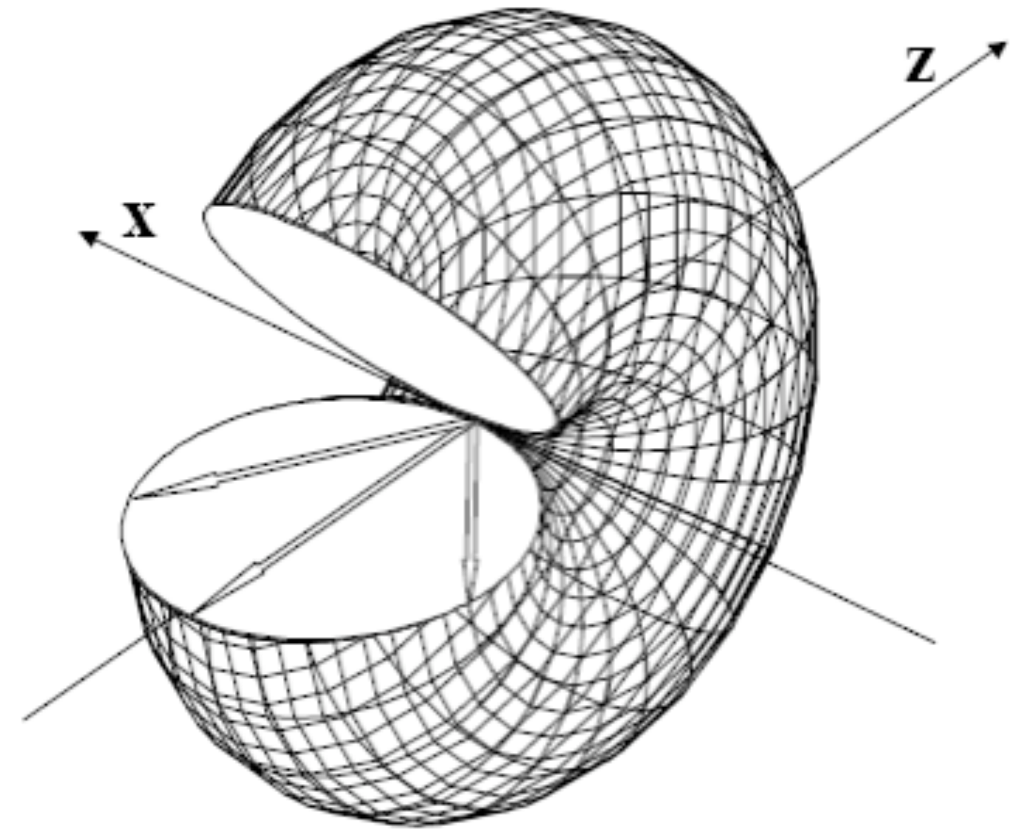
# Angular distribution of radiated power: non relativistic motion

Assuming $\bar{\beta} \approx \bar{0}$ and substituting the acceleration field

$$\bar{E}_{acc}(\bar{x},t) = \frac{e}{c}\left[\frac{\bar{n}\times(\bar{n}\times\dot{\bar{\beta}})}{R}\right]_{rit}$$

$$\frac{dP}{d\Omega} = \frac{c}{4\pi}\left|R\bar{E}_{acc}\right|^2 = \frac{e^2}{4\pi c}\left|\bar{n}\times(\bar{n}\times\dot{\bar{\beta}})\right|^2$$

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^2}\left|\dot{\bar{\beta}}\right|^2 \sin^2\theta$$




θ is the angle between the acceleration and the observation direction

Integrating over the angles gives the total radiated power

$$P = \frac{2}{3}\frac{e^2}{c}\left|\dot{\bar{\beta}}\right|^2$$ Larmor's formula

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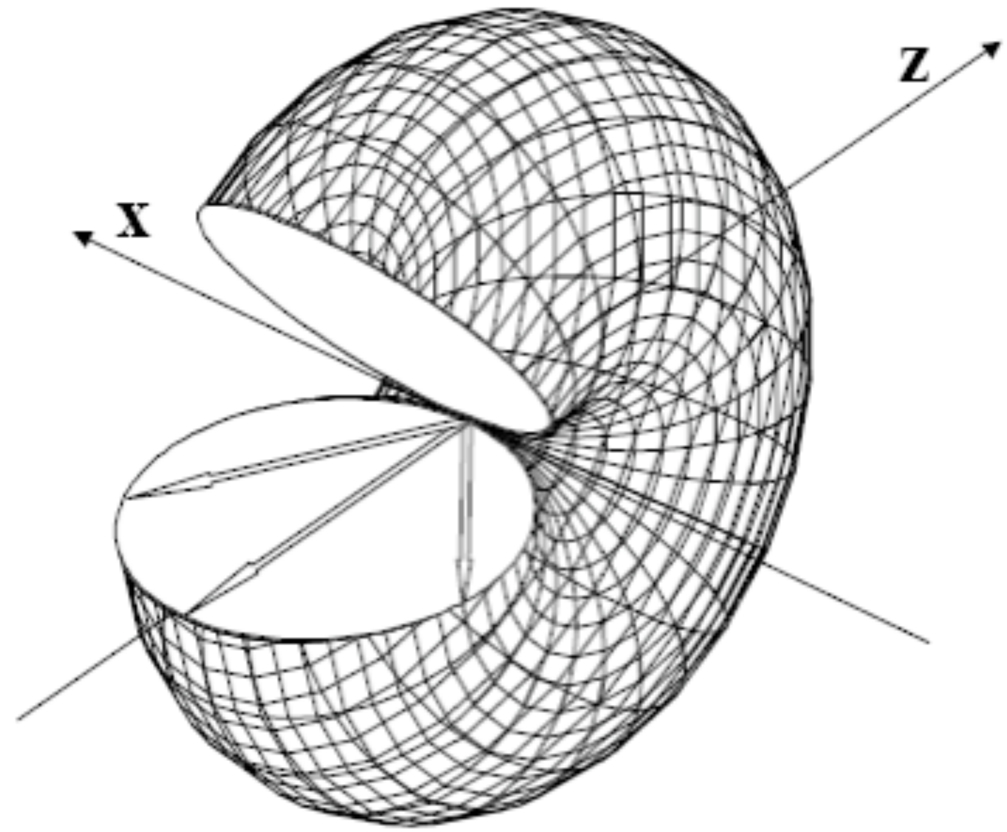


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Larmor's formula

polarization in the plane containing $\bar{n},\dot{\bar{\beta}}$

# Angular distribution of radiated power: relativistic motion

Substituting the acceleration field

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c} \frac{\left| \bar{n} \times \left[ (\bar{n} - \bar{\beta}) \times \dot{\bar{\beta}} \right] \right|^2}{(1 - \bar{n} \cdot \bar{\beta})^5}$$

The pattern depends on the details of velocity and acceleration but it is dominated by the denominator

Total radiated power: computed either by integration over the angles or by relativistic transformation of the 4-acceleration in Larmor's formula

$$P = \frac{2}{3} \frac{e^2}{c} \gamma^6 \left[ (\dot{\bar{\beta}})^2 - (\bar{\beta} \times \dot{\bar{\beta}})^2 \right]$$

Relativistic generalization of Larmor's formula

# Angular distribution of radiated power: relativistic motion

Substituting the acceleration field

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c} \frac{\left| \bar{n} \times \left[ (\bar{n} - \bar{\beta}) \times \dot{\bar{\beta}} \right] \right|^2}{(1 - \bar{n} \cdot \bar{\beta})^5}$$

emission is peaked in the direction of the velocity

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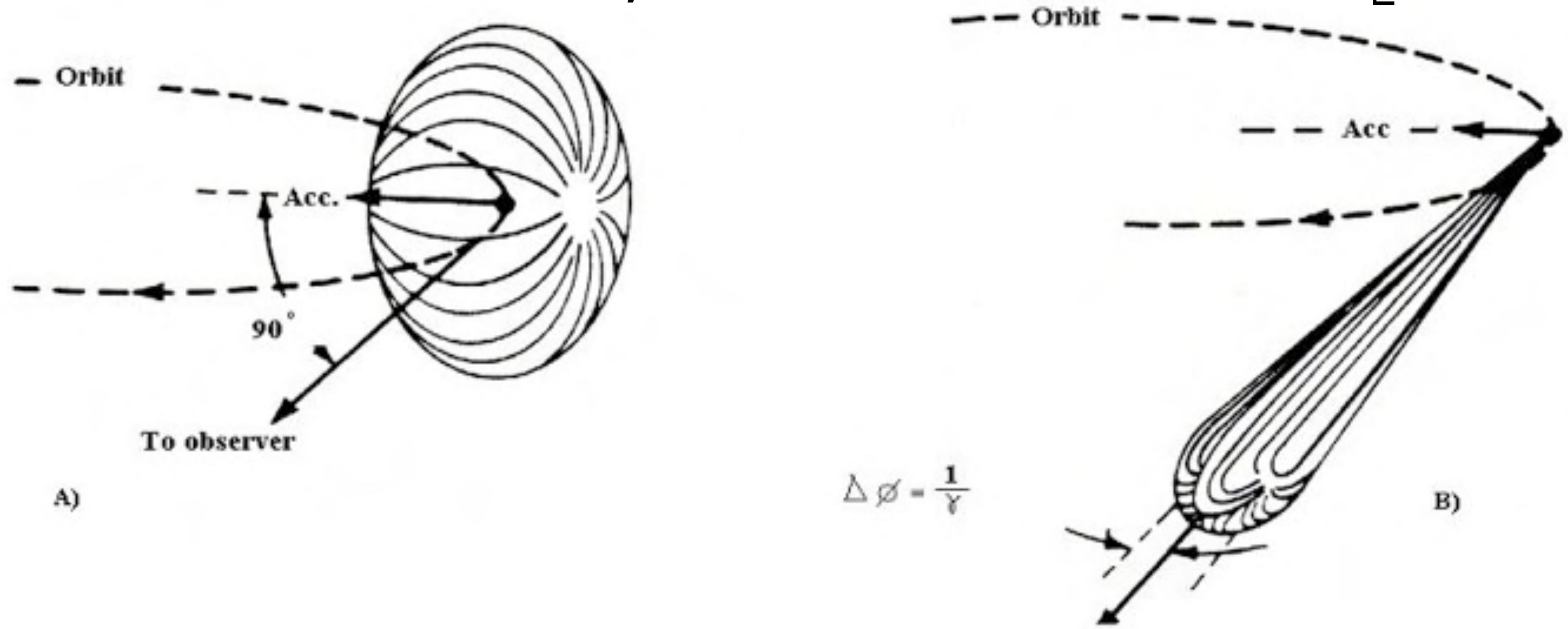
Relativistic generalization of Larmor's formula

# velocity ⊥ acceleration: synchrotron radiation

Assuming $\bar{\beta} \perp \dot{\bar{\beta}}$ and substituting the acceleration field

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c} \frac{\left| \bar{n} \times \left[ (\bar{n} - \bar{\beta}) \times \dot{\bar{\beta}} \right] \right|^2}{(1 - \bar{n} \cdot \bar{\beta})^5} = \frac{e^2}{4\pi c} \frac{\left| \dot{\bar{\beta}} \right|^2}{(1 - \beta \cos\theta)^3} \left[ 1 - \frac{\sin^2\theta \cos^2\phi}{\gamma^2 (1 - \beta\cos\theta)^2} \right]$$




Total radiated power

$$P = \frac{2}{3} \frac{e^2}{c} \left| \dot{\bar{\beta}} \right|^2 \gamma^4 \qquad P = \frac{2}{3} \frac{e^2}{m^2 c^3} \gamma^2 \left| \frac{d\bar{p}}{dt} \right|^2$$

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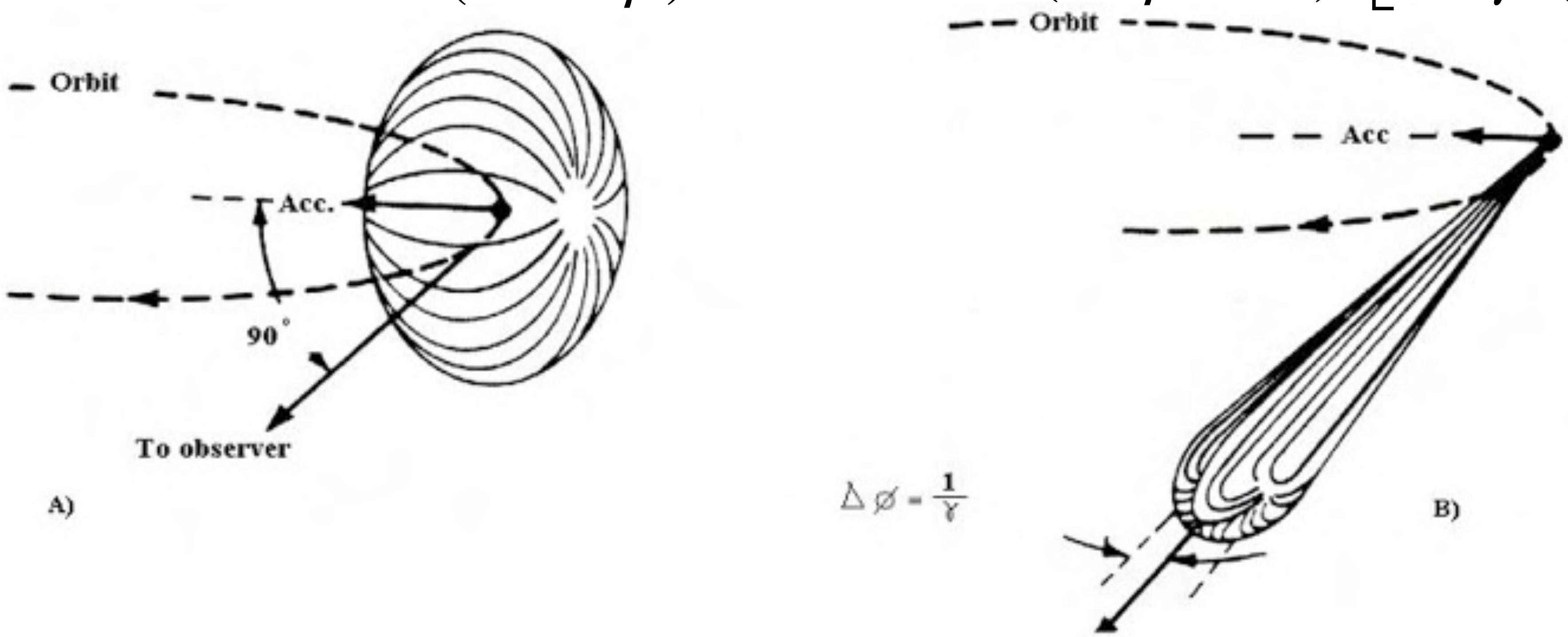


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Strong dependence $1/m^4$ on the rest mass

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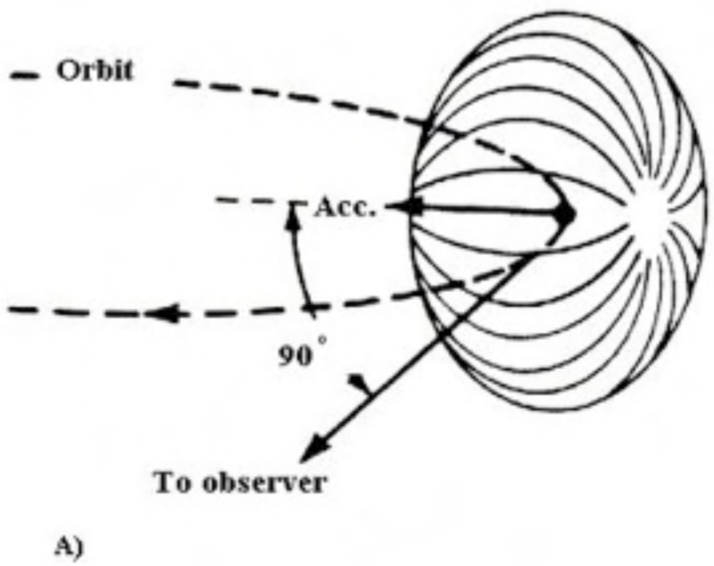


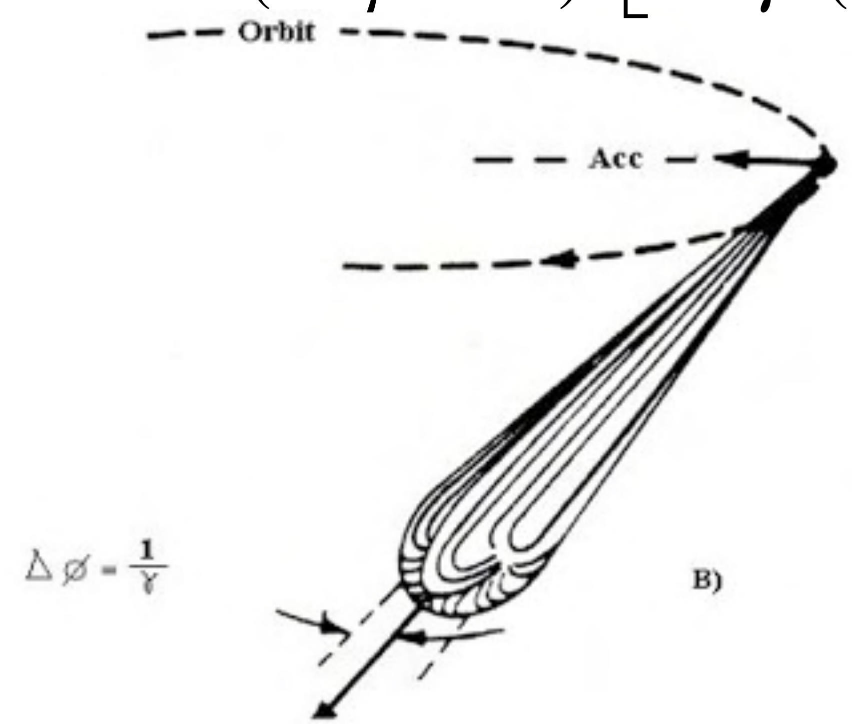


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Strong dependence $1/m^4$ on the rest mass

$P(v \perp a) \approx \gamma^2 P(v \parallel a)$

# The radiation integral

Angular and frequency distribution of the power received by an observer

$$\frac{d^2 I}{d\Omega d\omega} = 2\left|\bar{A}(\omega)\right|^2 = 2\frac{c}{4\pi}R^2\left|\hat{\bar{E}}(\omega)\right|^2$$

Neglecting the velocity fields and assuming the observer in the far field:
n constant

$$\frac{d^2 I}{d\Omega d\omega} = \frac{e^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \frac{\bar{n}\times\left[(\bar{n}-\bar{\beta})\times\dot{\bar{\beta}}\right]}{(1-\bar{n}\cdot\bar{\beta})^2} e^{i\omega(t-\bar{n}\cdot\bar{r}(t)/c)}dt\right|^2$$ Radiation Integral

and since $$\frac{\bar{n}\times\left[(\bar{n}-\bar{\beta})\times\dot{\bar{\beta}}\right]}{(1-\bar{n}\cdot\bar{\beta})^2} = \frac{d}{dt}\left[\frac{\bar{n}\times\left(\bar{n}\times\bar{\beta}\right)}{1-\bar{n}\cdot\bar{\beta}}\right]$$

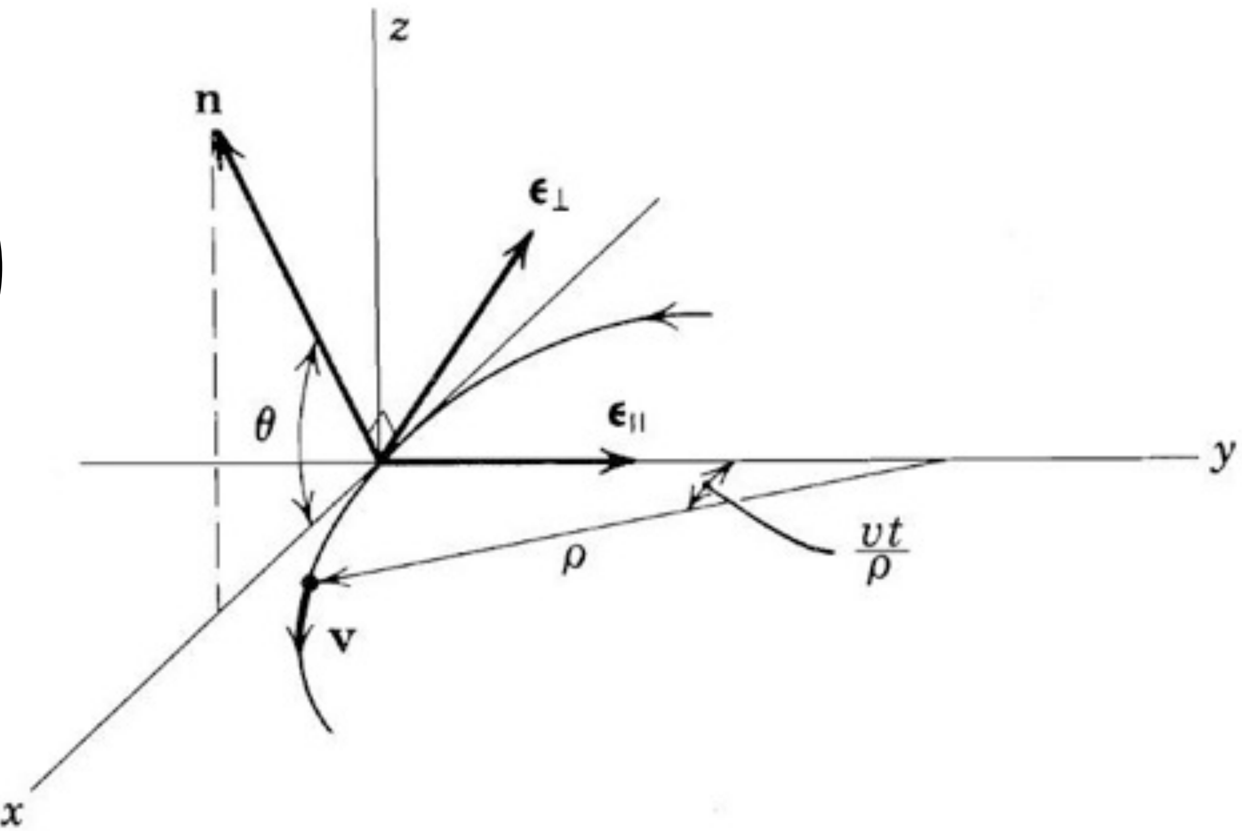
we can integrate by parts and obtain: $$\frac{d^2 I}{d\Omega d\omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \bar{n}\times\left(\bar{n}\times\bar{\beta}\right) e^{i\omega(t-\bar{n}\cdot\bar{r}(t)/c)}dt\right|^2$$

- determine the particle motion
- compute the cross products and the phase factor
- integrate each component and take the vector square modulus

# Radiation integral for synchrotron radiation

Trajectory of the arc of circumference

$$\bar{r}(t) = \left(\rho\left(1-\cos\frac{\beta c}{\rho}t\right), \rho\left(\sin\frac{\beta c}{\rho}t\right), 0\right)$$




In the limit of small angles we compute

$$\bar{n}\times(\bar{n}\times\bar{\beta}) = \beta\left[-\bar{\varepsilon}_{\parallel}\sin\left(\frac{\beta ct}{\rho}\right)+\bar{\varepsilon}_{\perp}\cos\left(\frac{\beta ct}{\rho}\right)\sin\theta\right]$$

$$\omega\left(t-\frac{\bar{n}\cdot\bar{r}(t)}{c}\right) = \omega\left[t-\frac{\rho}{c}\sin\left(\frac{\beta ct}{\rho}\right)\cos\theta\right]$$

Substituting into the radiation integral and introducing $\xi = \frac{\rho\omega}{3c\gamma^3}\left(1+\gamma^2\theta^2\right)^{3/2}$

$$\frac{d^2 I}{d\Omega\, d\omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c\gamma^2}\right)^2\left(1+\gamma^2\theta^2\right)^2\left[K_{2/3}^2(\xi)+\frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}K_{1/3}^2(\xi)\right]$$

# Polarisation of synchrotron radiation

$$\frac{d^2 I}{d\Omega\, d\omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c\gamma^2}\right)^2 \left(1+\gamma^2\theta^2\right)^2 \left[K_{2/3}^2(\xi) + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2} K_{1/3}^2(\xi)\right]$$

In the orbit plane θ = 0, the polarisation is purely horizontal

Angular distribution of the energy radiated

$$\frac{dI}{d\Omega} = \int_0^\infty \frac{d^2 I}{d\omega\, d\Omega} d\omega = \frac{7}{16}\frac{e^2\gamma^5}{\rho}\frac{1}{\left(1+\gamma^2\theta^2\right)^{5/2}}\left[1+\frac{5}{7}\frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\right]$$

Integrating on all the angles we get a polarization on the plan of the orbit 7 times larger than on the plan perpendicular to the orbit

# Polarisation of synchrotron radiation

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Polarisation in the orbit plane

In the orbit plane θ = 0, the polarisation is purely horizontal

Angular distribution of the energy radiated

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Polarisation in the orbit plane

Polarisation orthogona
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Integrating on all the angles we get a polarization on the plan of the orbit 7 times larger than on the plan perpendicular to the orbit

# Critical frequency and critical angle

$$\frac{d^2 I}{d\Omega\, d\omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c\gamma^2}\right)^2 \left(1+\gamma^2\theta^2\right)^2 \left[K_{2/3}^2(\xi) + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2} K_{1/3}^2(\xi)\right]$$

The radiation intensity is negligible for ξ >> 1

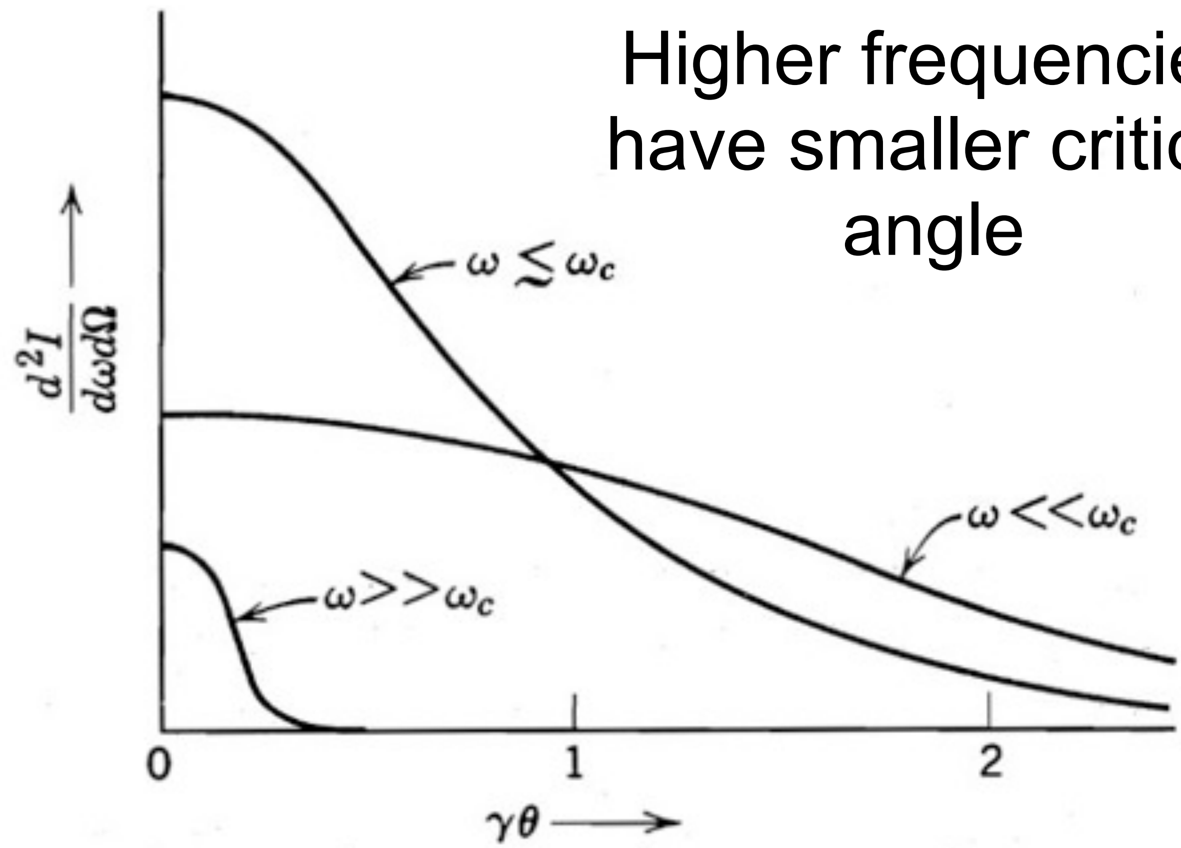
$$\xi = \frac{\omega\rho}{3c\gamma^3}\left(1+\gamma^2\theta^2\right)^{3/2} \gg 1$$

Critical frequency

$$\omega \gg \frac{3c\gamma^3}{\rho\,(1+\gamma^2\theta^2)^{3/2}} \qquad \omega_c = \frac{3}{2}\frac{c}{\rho}\gamma^3$$

Critical angle

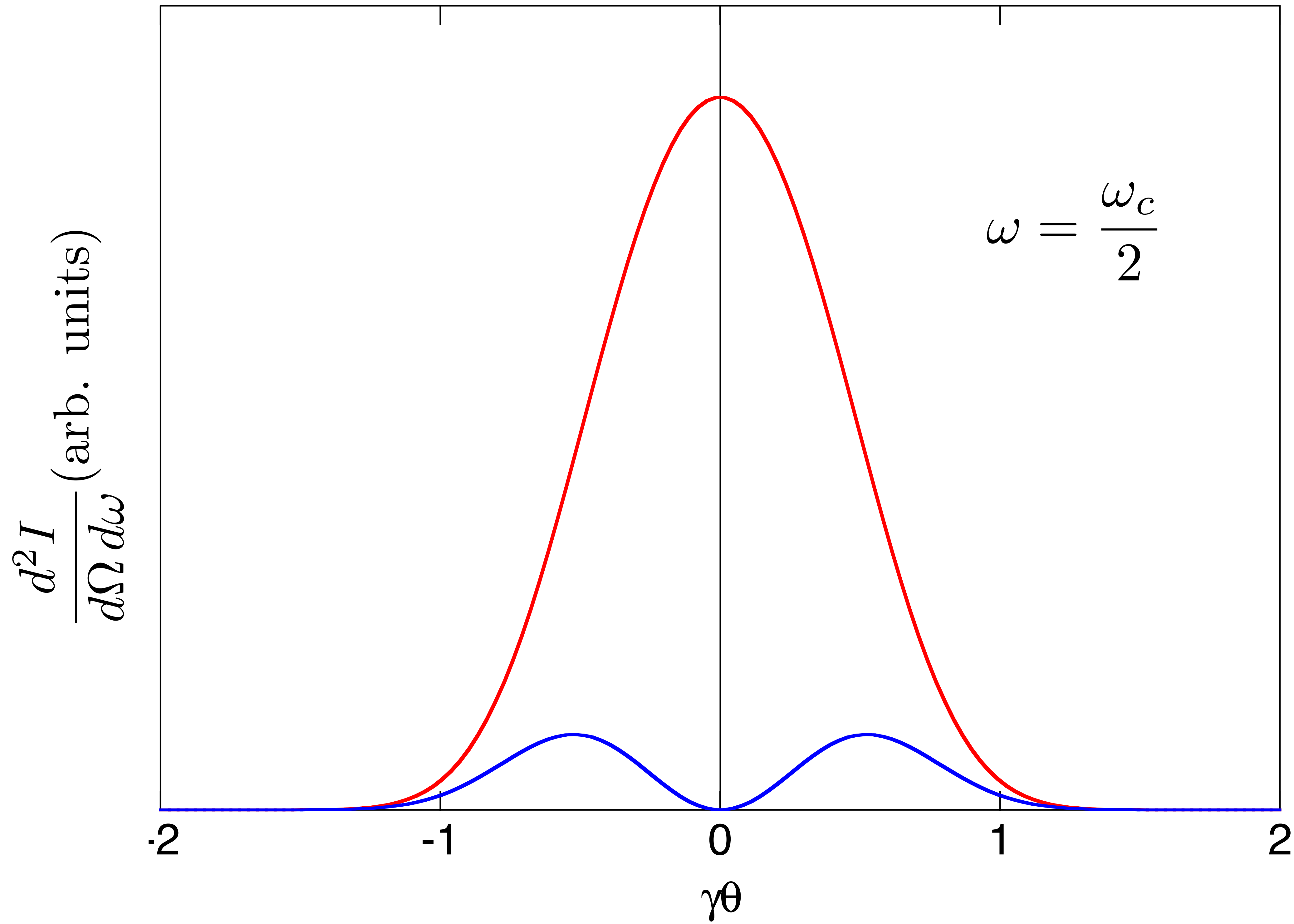
$$\theta \gg \left(\frac{3c}{\omega\rho}\right)^{1/3} \qquad \theta_c = \frac{1}{\gamma}\left(\frac{\omega_c}{\omega}\right)^{1/3}$$




Higher frequencies have smaller critical angle

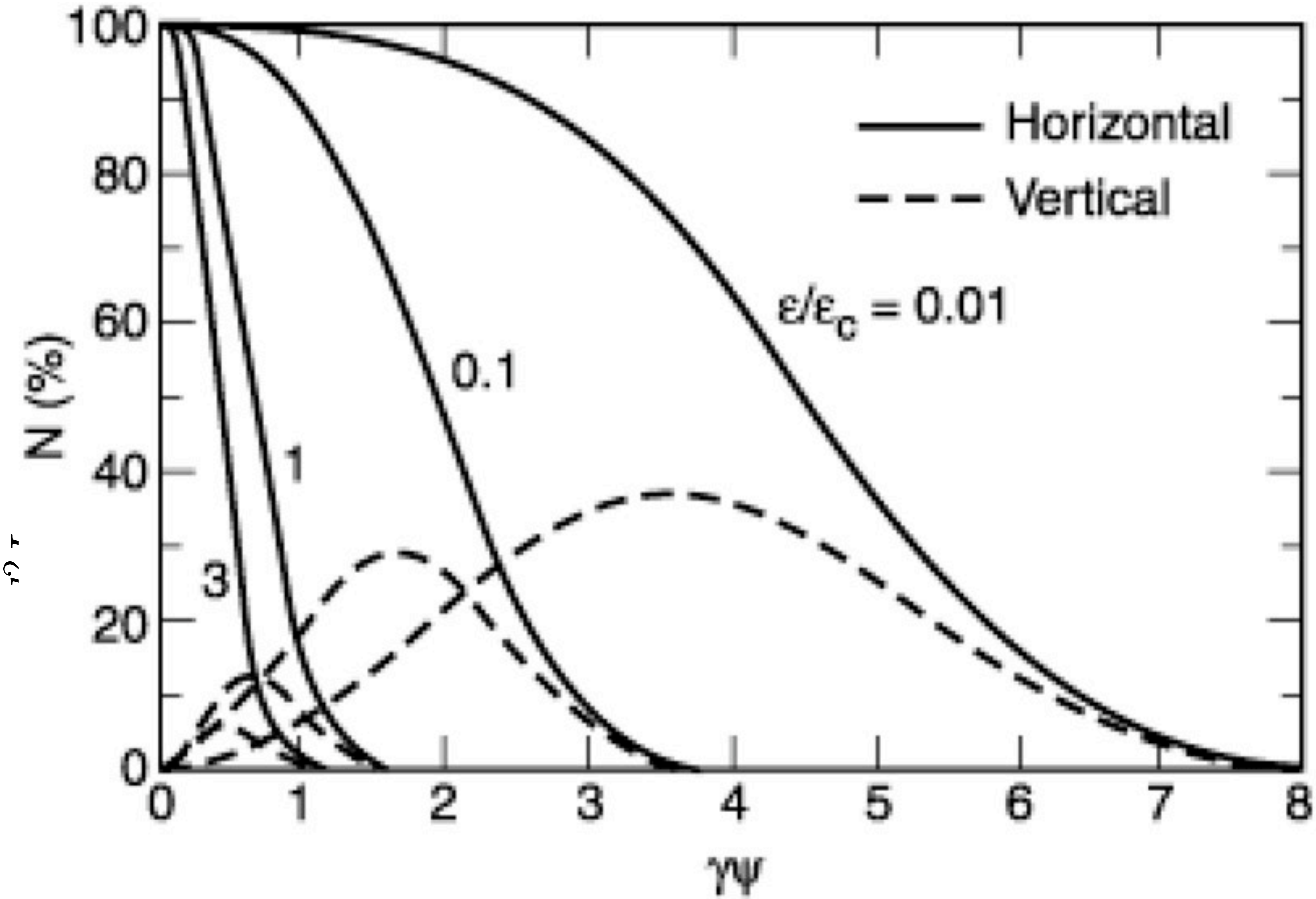
# Polarization

$$\frac{d^2 I}{d\Omega\, d\omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c\gamma^2}\right)^2\left[\left(1+\gamma^2\theta^2\right)^2 K_{2/3}^2(\xi) + \left(1+\gamma^2\theta^2\right)\gamma^2\theta^2 K_{1/3}^2(\xi)\right]$$



# Polarization

$$\frac{d^2 I}{d\Omega\, d\omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c\gamma^2}\right)^2 \left(1+\gamma^2\theta^2\right)^2 \left[K_{2/3}^2(\xi) + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2} K_{1/3}^2(\xi)\right]$$



# Frequency distribution of radiated energy

Integrating on all angles we get the frequency distribution of the energy radiated

$$\frac{dI}{d\omega} = \sqrt{3}\frac{e^2}{c}\gamma\frac{\omega}{\omega_C}\int_{\omega/\omega_C}^{\infty} K_{5/3}(x)dx$$

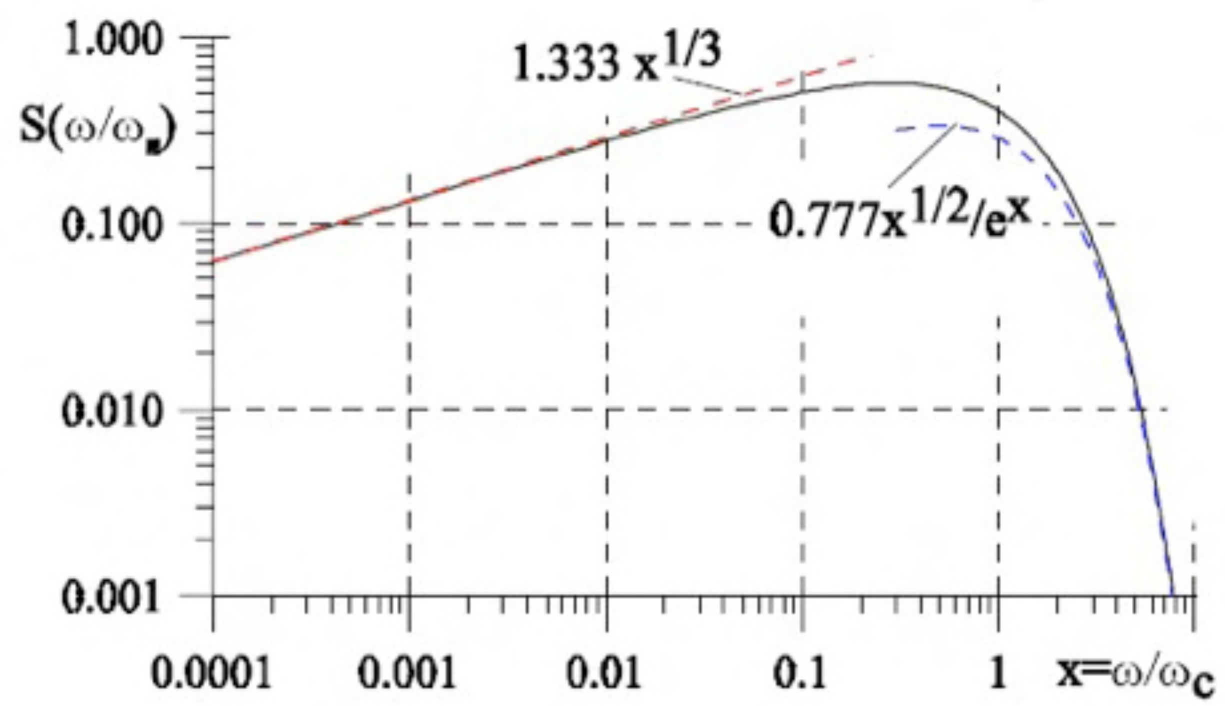
$$\frac{dI}{d\omega} \approx \frac{e^2}{c}\left(\frac{\omega\rho}{c}\right)^{1/3} \qquad \omega << \omega_c$$

$$\frac{dI}{d\omega} \approx \sqrt{\frac{3\pi}{2}\frac{e^2}{c}\gamma\left(\frac{\omega}{\omega_c}\right)^{1/2}} e^{-\omega/\omega_c} \qquad \omega >> \omega_c$$

often expressed in terms of the function S(ξ) with $\xi = \omega/\omega_c$

$$S(\xi) = \frac{9\sqrt{3}}{8\pi}\xi\int_{\xi}^{\infty} K_{5/3}(x)dx \qquad \int_0^{\infty} S(\xi)d\xi = 1$$

$$\frac{dI}{d\omega} = \sqrt{3}\frac{e^2}{c}\gamma\frac{\omega}{\omega_C}\int_{\omega/\omega_C}^{\infty} K_{5/3}(x)dx = \frac{8\pi e^2\gamma}{9c}S(\xi)$$



# Total power radiated via synchrotron radiation emission in a storage ring

Total radiated power

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\gamma^2\left|\frac{d\overline{p}}{dt}\right|^2 = \frac{2}{3}e^2c\frac{\gamma^4}{\rho^2}$$

In the time spent in the bendings the particle loses the energy $U_0$

$$U_0 = \int P dt = PT_b = P\frac{2\pi\rho}{c}$$

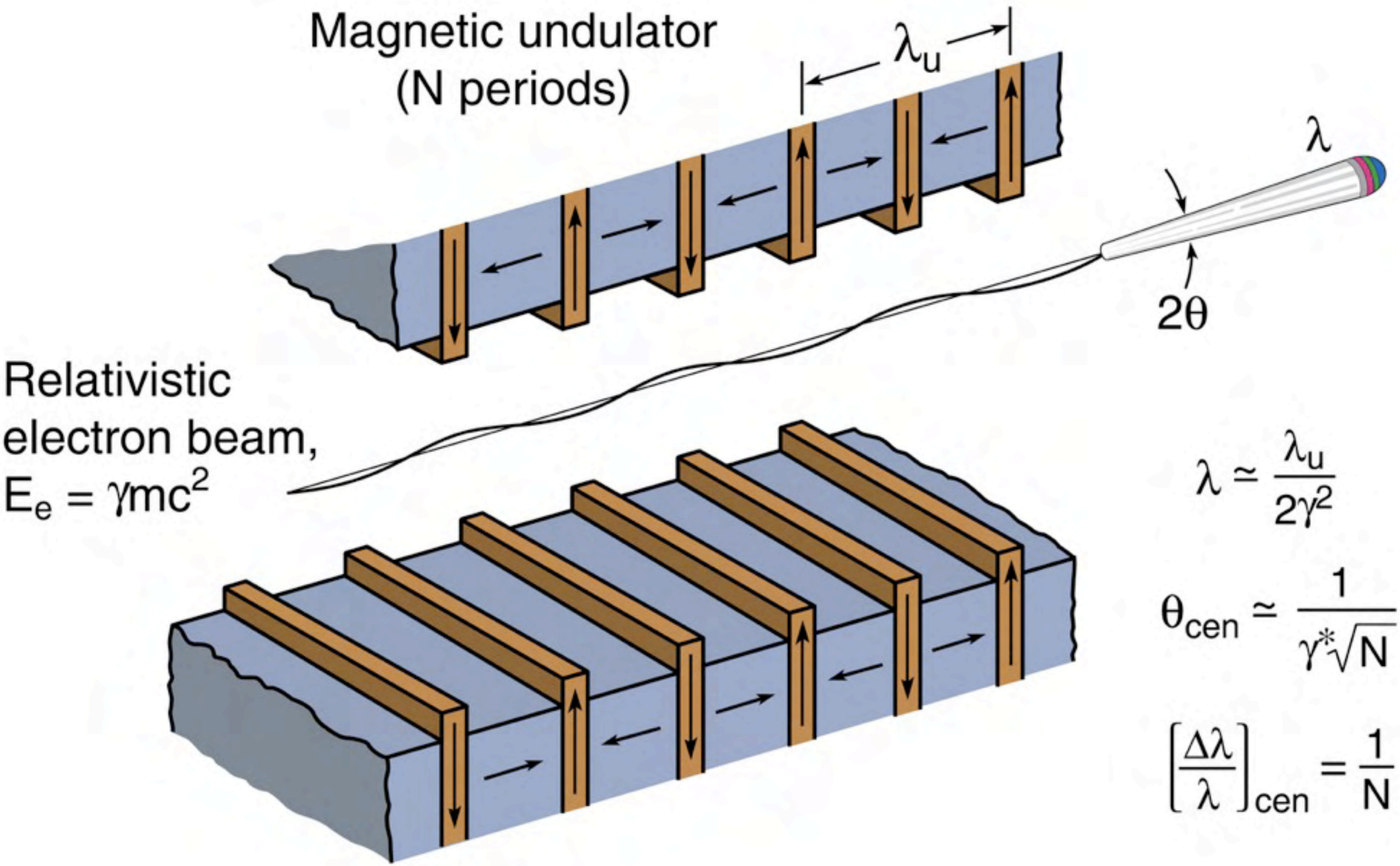
Energy losses per turn

$$U_0(eV) = \frac{e^2\gamma^4}{3\varepsilon_0\rho} = 88462.7\frac{E(GeV)^4}{\rho(m)}$$

One can verify that

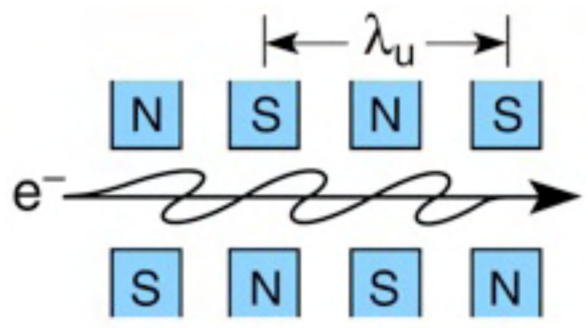
$$P = \frac{U_0}{T_b} = \frac{1}{T_b}\int_0^{\omega}\frac{dI}{d\omega}d\omega = \frac{1}{T_b}\frac{2e^2\gamma}{9\varepsilon_0 c}\omega_c\int_0^{\omega}\xi d\xi\int_{\xi}^{\infty}K_{5/3}(x)dx = \frac{e^2c}{6\varepsilon_0 c}\frac{\gamma^4}{\rho^2}$$

# Undulator radiation



# Undulator radiation

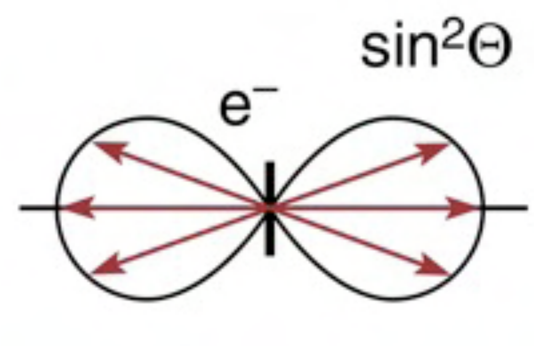
**Laboratory Frame of Reference**




$$E = \gamma m c^2$$

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

**Frame of Moving Electron**



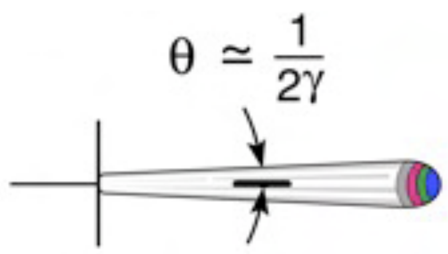

the electron radiates at the Lorentz contracted wavelength

$$\lambda' = \frac{\lambda_u}{\gamma}$$

Bandwidth:

$$\frac{\Delta\lambda'}{\lambda'} = \frac{1}{N}$$

**Frame of Observer**




Doppler shortened wavelength:

$$\lambda = \lambda' \gamma \left(1 - \beta \cos\theta\right)$$

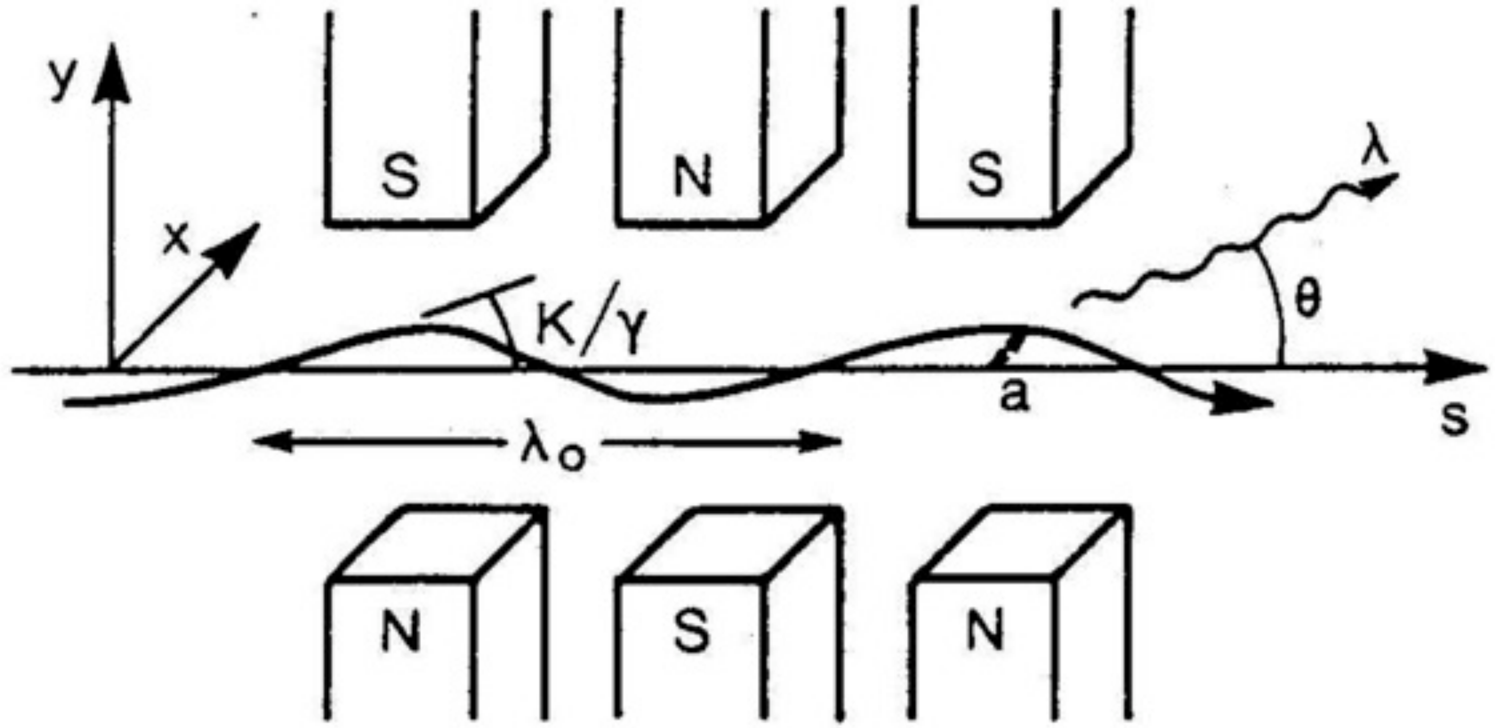
$$\lambda \simeq \frac{\lambda_u}{2\gamma^2}\left(1 + \gamma^2\theta^2\right)$$

and considering the transverse motion

$$\lambda \simeq \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

$$K = \frac{e B_0 \lambda_u}{2\pi m_0 c}$$

# Undulator radiation




Periodic array of magnetic poles providing a sinusoidal magnetic field on axis:

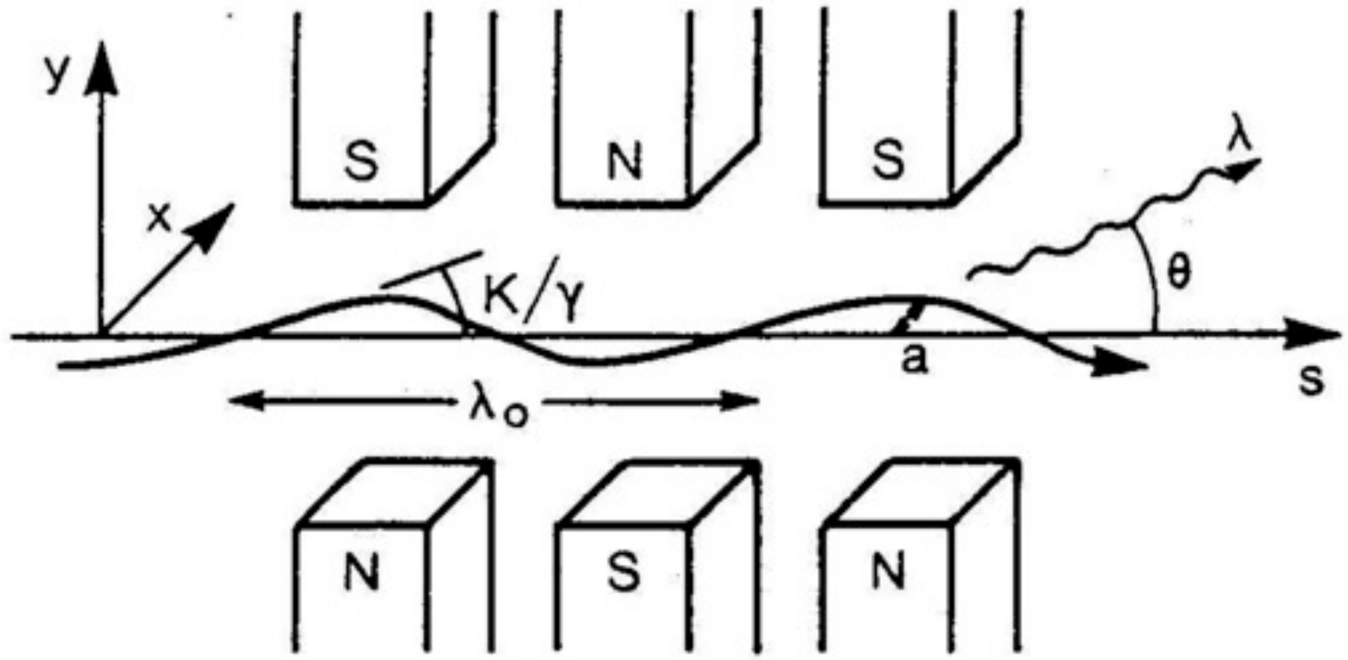
$$B_y = B_0 \sin\left(\frac{2\pi z}{\lambda_0}\right) = B_0 \sin(kz)$$

The Lorentz force is: $\vec{F} = \gamma m \vec{a} = -e\vec{v} \times \vec{B}$

So we get the set of differential equations:

$$\begin{cases} \ddot{x} = \frac{e}{\gamma m}\left(-\dot{z}B_y\right) \\ \ddot{z} = \frac{e}{\gamma m}\left(\dot{x}B_y\right) \end{cases}$$

# Undulator radiation




$$\begin{cases} \ddot{x} = \frac{e}{\gamma m}\left(-\dot{z}B_y\right) \\ \ddot{z} = \frac{e}{\gamma m}\left(\dot{x}B_y\right) \end{cases}$$

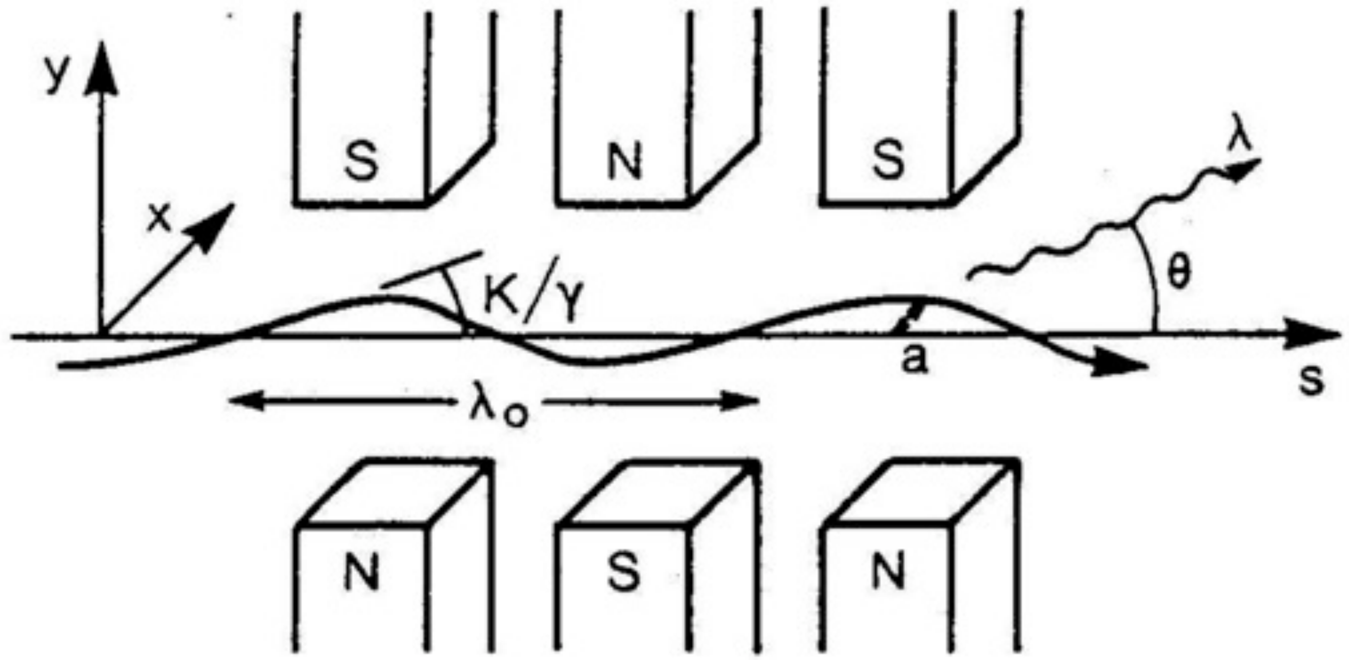
integration of the first equation gives:

$$\dot{x} = \frac{eB_0}{\gamma m}\frac{\cos\left(kz\right)}{k} \qquad \beta_x = \frac{\dot{x}}{c} = \frac{K}{\gamma}\cos\left(kz\right)$$

where we have defined

$$K = \frac{eB_0\lambda_0}{2\pi mc} \cong 0.9337 B_0\,[\mathrm{T}]\,\lambda_0\,[\mathrm{cm}]$$

# Undulator radiation

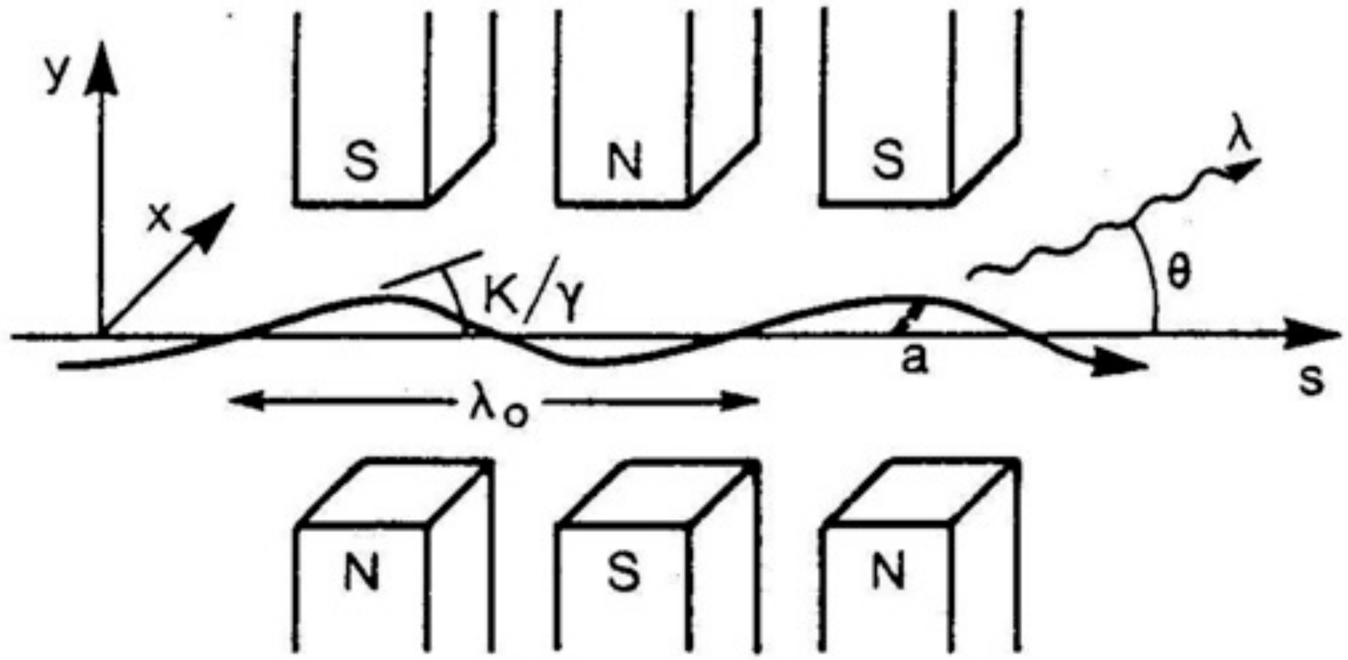



$$\beta_x = \frac{\dot{x}}{c} = \frac{K}{\gamma}\cos(kz)$$

The horizontal motion of the electron causes the electron velocity along the z axis to vary also, since the electron energy, and hence total speed remain unaltered:

$$\beta_x^2 + \beta_z^2 = \beta^2 \ (=\text{constant})$$

# Undulator radiation



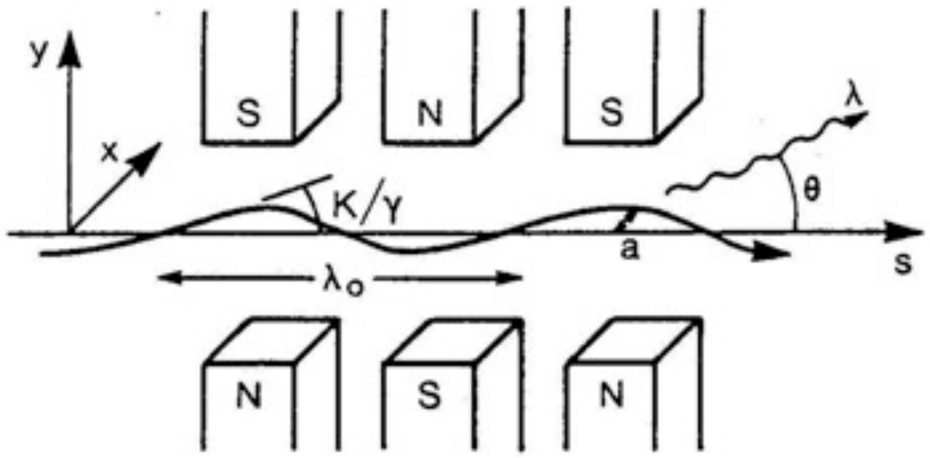

$$\beta_x^2 + \beta_z^2 = \beta^2 \ (=\text{constant})$$

$$\beta_z = \sqrt{\beta^2 - \beta_x^2} = \sqrt{\beta^2 - \left(\frac{K}{\gamma}\cos(kz)\right)^2} =$$

$$= \beta\sqrt{1 - \frac{K^2}{\gamma^2\beta^2}\cos^2(kz)} =$$

$$\simeq \beta\left(1 - \frac{K^2}{4\gamma^2} - \frac{K^2}{4\gamma^2}\cos 2kz\right)$$

# Undulator radiation




The average velocity along the z-axis is thus:

$$\langle \beta_z \rangle \simeq \beta \left(1 - \frac{K^2}{4\gamma^2}\right)$$

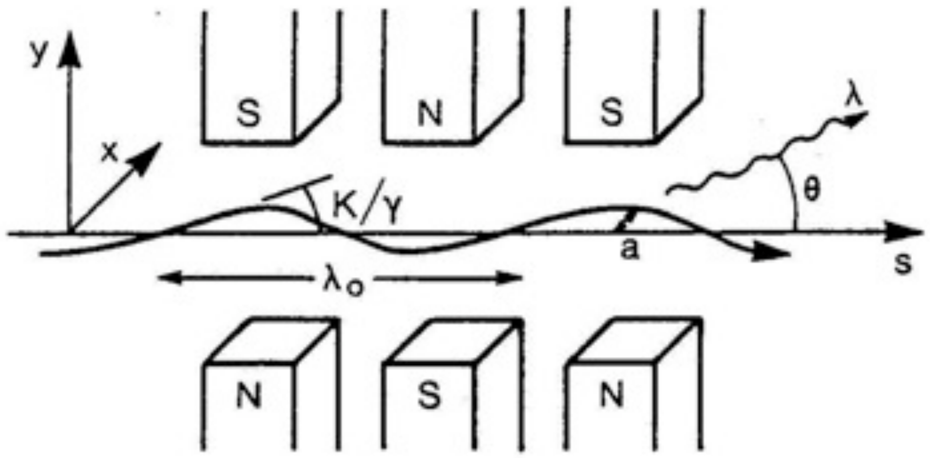
Since K/γ<<1, we can approximate z in the argument of the cosine with <β>ct so:

$$\dot{x} = \frac{K}{\gamma} c \cos \Omega t$$

$$\Omega = \frac{2\pi \langle \beta \rangle c}{\lambda_0}$$

$$\dot{z} = \langle \beta \rangle c - \frac{K^2}{4\gamma^2} c \cos 2\Omega t$$

# Undulator radiation




$$\dot{x} = \frac{K}{\gamma} c \cos \Omega t$$

$$\dot{z} = \langle \beta \rangle c - \frac{K^2}{4\gamma^2} c \cos 2\Omega t \qquad \Omega = \frac{2\pi \langle \beta \rangle c}{\lambda_0}$$

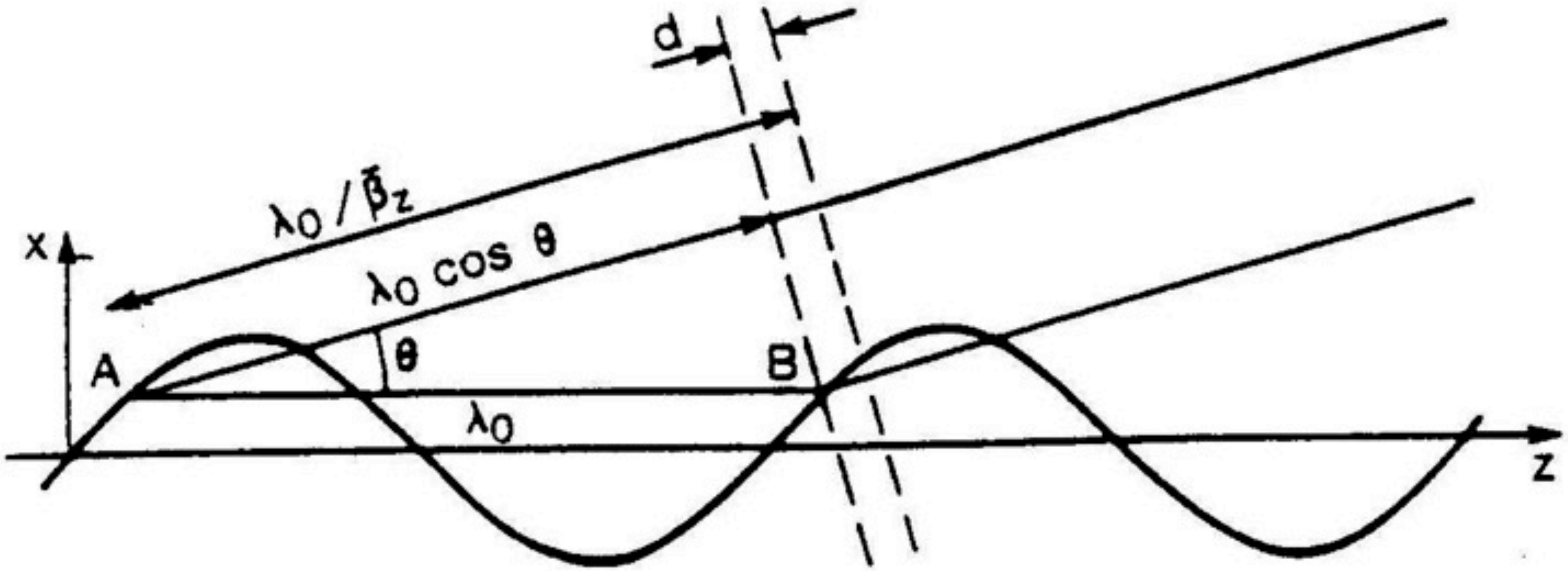
which can be integrated directly to give:

$$x = \frac{K}{\gamma} \frac{c}{\Omega} \sin \Omega t = \frac{K}{\gamma} \frac{\lambda_0}{2\pi \langle \beta \rangle} \sin \Omega t$$

$$z = \langle \beta \rangle ct - \frac{K^2}{4\gamma^2} \frac{\lambda_0}{4\pi \langle \beta \rangle} \sin 2\Omega t$$

The actual motion of the particle is quite small: for example, a realistic device with a 50 mm period and K = 2 in a 2 GeV ring has a maximum deflection angle (x′) of 0.5 mrad and oscillation amplitude of 4 μm. The z-motion is even smaller with an amplitude of only 2.6 Å.

# Undulator radiation




**Interference**

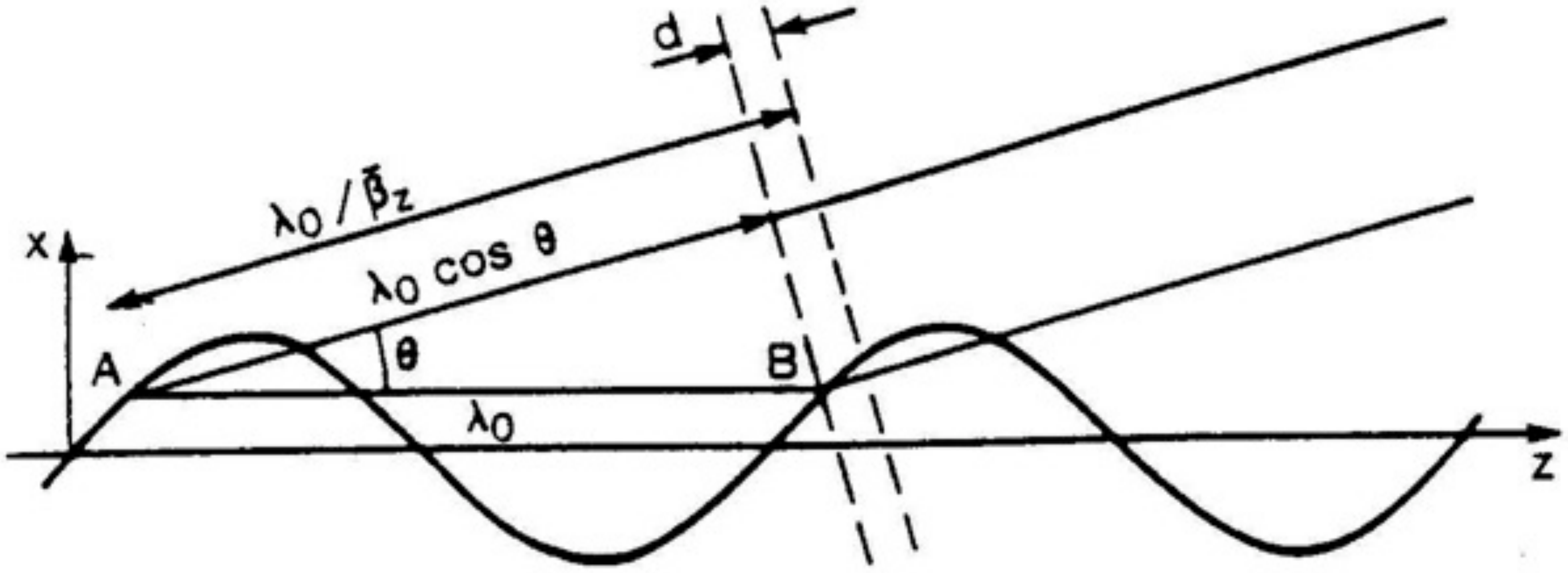
The difference in optical paths between the radiation emitted at A and the radiation emitted at B at an angle θ is

$$d = \lambda_0 \left( \frac{1}{\langle \beta \rangle} - \cos\theta \right)$$

and we get constructive inteference if d=nλ

$$\lambda = \frac{\lambda_0}{2n\gamma^2} \left( 1 + \frac{K^2}{2} + \gamma^2 \theta^2 \right)$$

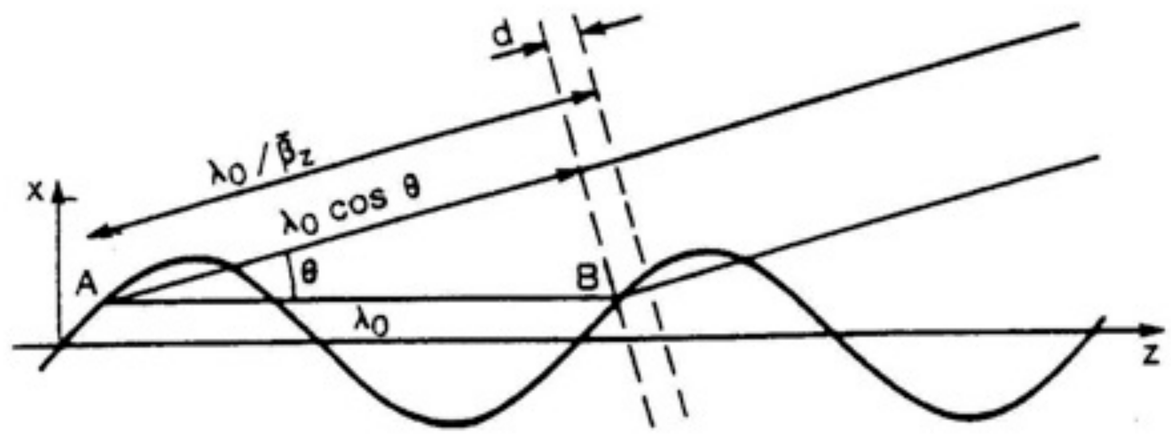
# Undulator radiation




$$\lambda = \frac{\lambda_0}{2n\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

- The fundamental wavelength of the radiation is very much shorter than the period length of the device, because of the large $\gamma^2$ term (for electrons, γ = 1957 E [GeV])
- The wavelength of the harmonics can be varied either by changing the electron beam energy (γ) or the insertion device field strength, and hence K value.
- The wavelength varies with observation angle. Overall therefore the spectrum covers a wide range of wavelength. However, if the range of observation angles is restricted using a "pinhole" aperture, the spectrum will show a series of lines at harmonic frequencies.

# Undulator radiation




The constructive interference condition over the whole length for an undulator of length L an d N periods gives:

$$\frac{L}{\langle\beta\rangle} - L\cos\theta = nN\lambda$$

Destructive interference is obtained for a wavelength which satisfies:

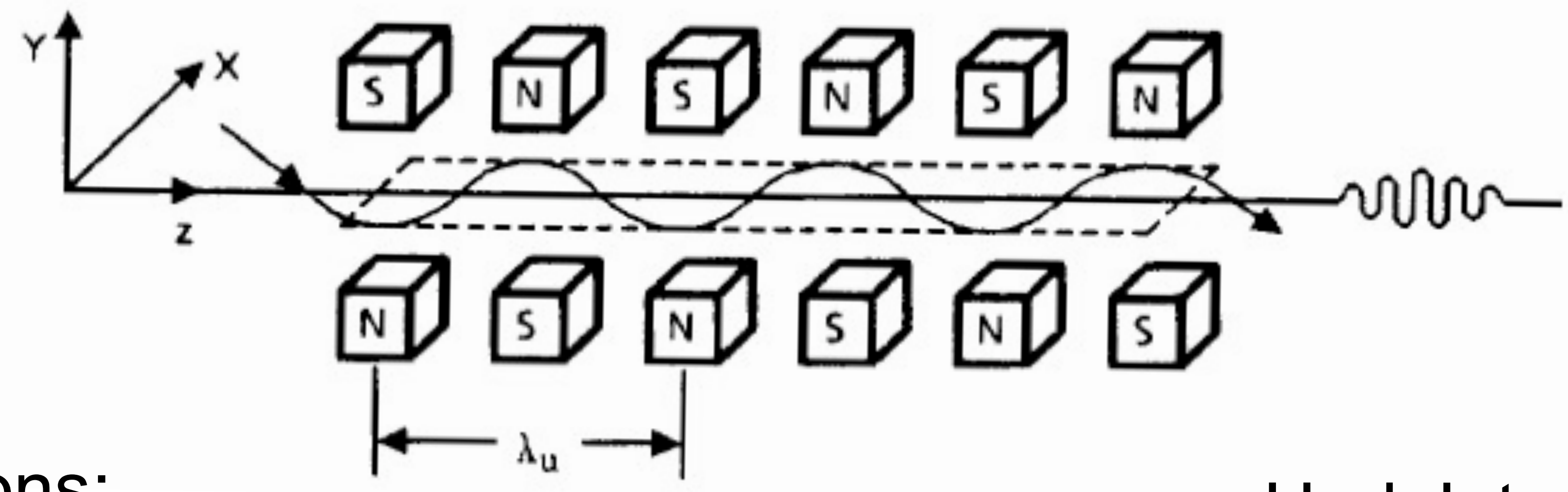
$$\frac{L}{\langle\beta\rangle} - L\cos\theta = nN\lambda' + \lambda'$$

Therefore: $$\frac{\Delta\lambda}{\lambda} = \frac{1}{nN}$$

and for the angular aperture we get: $$\Delta\theta = \sqrt{\frac{2\lambda}{L}} = \frac{1}{\gamma}\sqrt{\frac{1+\frac{K^2}{2}}{nN}}$$

# Undulators and wigglers

Periodic array of magnetic poles providing a sinusoidal magnetic field on axis:




$$B = (0, \quad B_0 \sin(k_u z), \quad 0,)$$

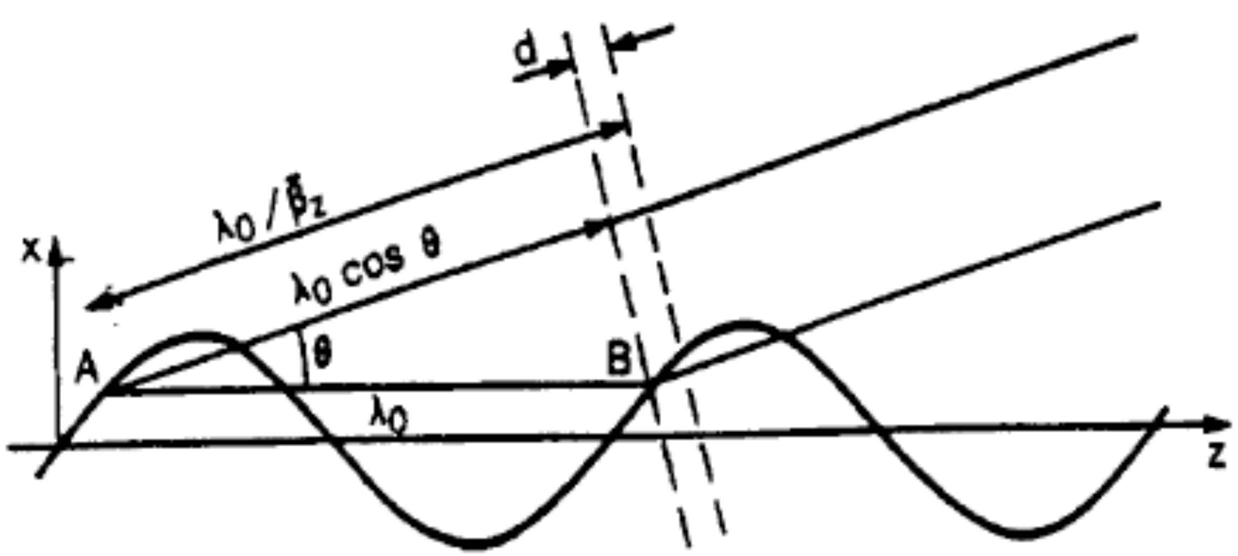
Solution of equation of motions:

$$\bar{r}(t) = -\frac{\lambda_u K}{2\pi\gamma} \sin \omega_u t \cdot \hat{x} + \left( \bar{\beta}_z ct + \frac{\lambda_u K^2}{16\pi\gamma^2} \cos(2\omega_u t) \right) \cdot \hat{z}$$

$$K = \frac{eB_0 \lambda_u}{2\pi mc}$$ Undulator parameter

$$\bar{\beta}_z = 1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

Constructive interference of radiation emitted at different poles



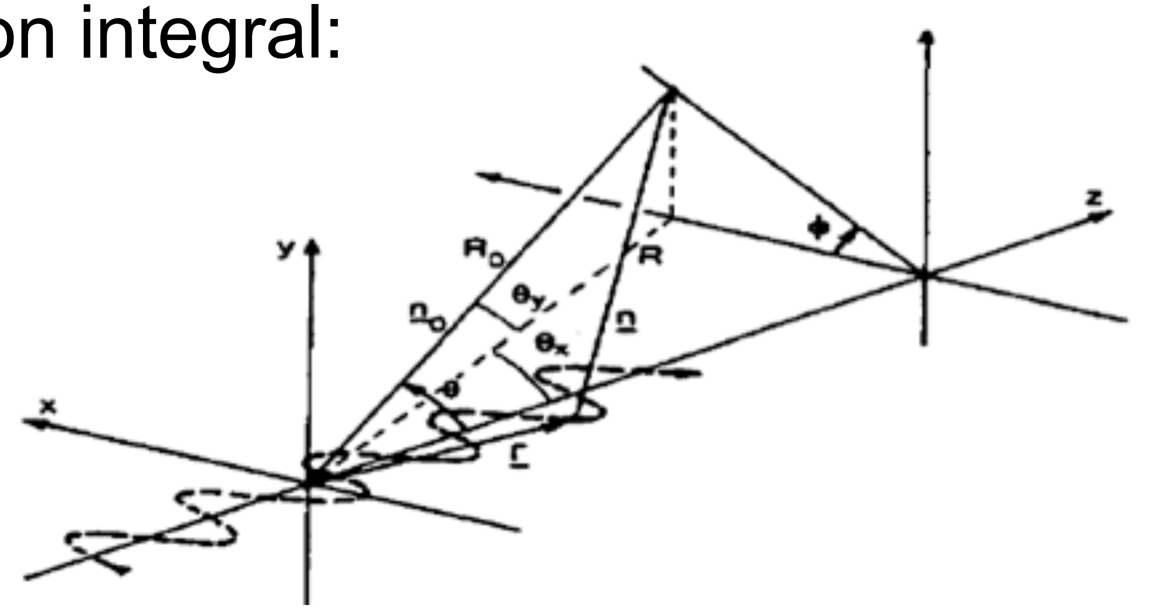

$$d = \frac{\lambda_u}{\bar{\beta}} - \lambda_u \cos\theta = n\lambda$$

$$\lambda_n = \frac{\lambda_u}{2\gamma^2 n}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

# Radiation integral for a linear undulator (I)

The angular and frequency distribution of the energy emitted by a wiggler is computed again with the radiation integral:

$$\frac{d^2 I}{d\Omega d\omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{n}\times(\hat{n}\times\bar{\beta})e^{i\omega(t-\hat{n}\cdot\bar{r}/c)}dt\right|^2$$




Using the periodicity of the trajectory

$$\frac{d^2 I}{d\Omega d\omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\lambda_0/2\bar{\beta}c}^{\lambda_0/2\bar{\beta}c} \hat{n}\times(\hat{n}\times\bar{\beta})e^{i\omega(t-\hat{n}\cdot\bar{r}/c)}dt\right|^2 \left|1+e^{i\delta}+e^{i2\delta}+...+e^{i(N-1)\delta}\right|^2 \quad \delta = \frac{2\pi\omega}{\omega_{res}(\theta)}$$

$$L\left(N\frac{\Delta\omega}{\omega_{res}(\theta)}\right) = \frac{\sin^2(N\pi\Delta\omega/\omega_{res})}{N^2\sin^2(\pi\Delta\omega/\omega_{res})} \quad F_n(K,\theta,\phi) \propto \left|\int_{-\lambda_0/2\bar{\beta}c}^{\lambda_0/2\bar{\beta}c} \hat{n}\times(\hat{n}\times\bar{\beta})e^{i\omega(t-\hat{n}\cdot\bar{r}/c)}dt\right|^2$$

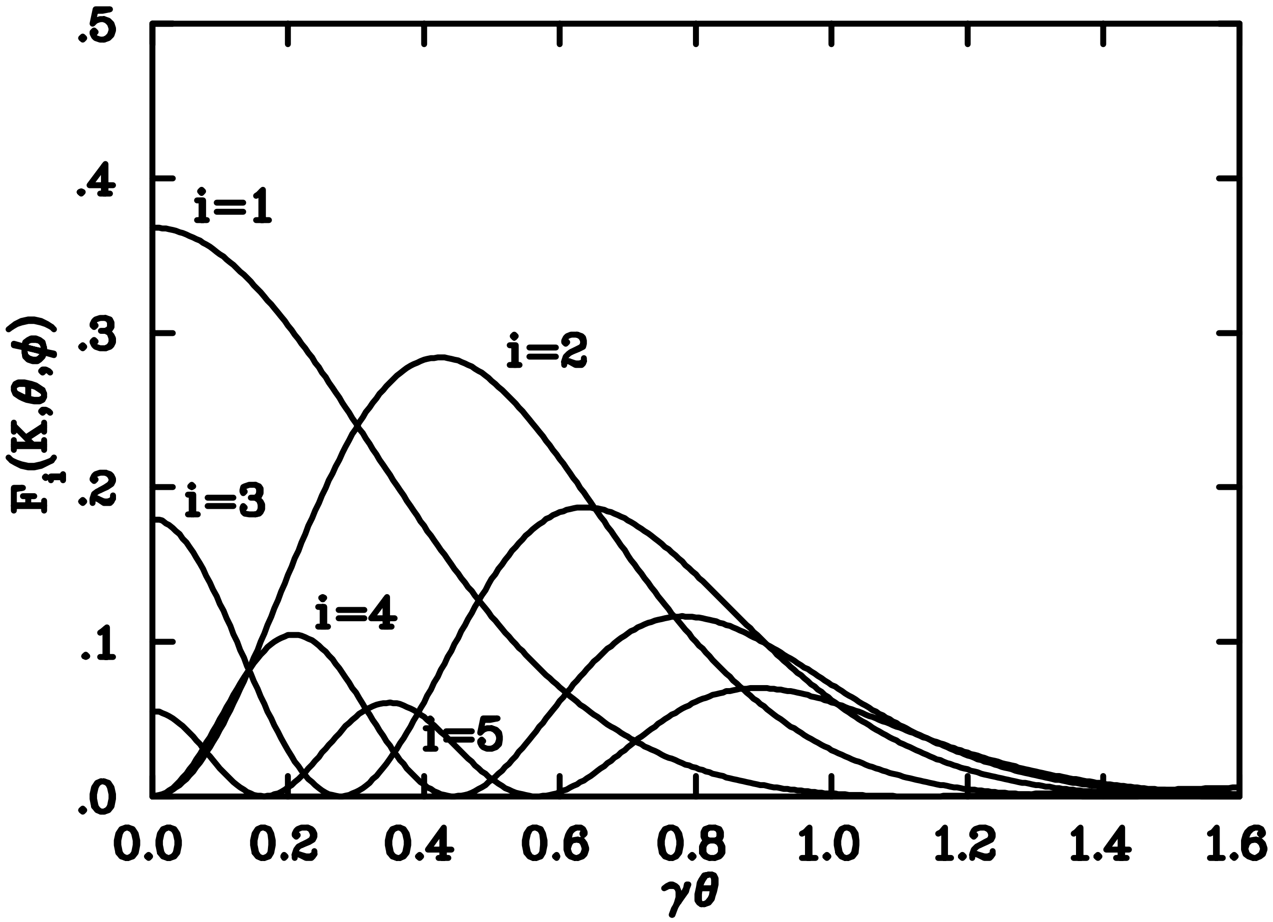
$$\frac{d^2 I}{d\Omega d\omega} = \frac{e^2\gamma^2 N^2}{c} L\left(N\frac{\Delta\omega}{\omega_{res}(\theta)}\right)F_n(K,\theta,\phi)$$

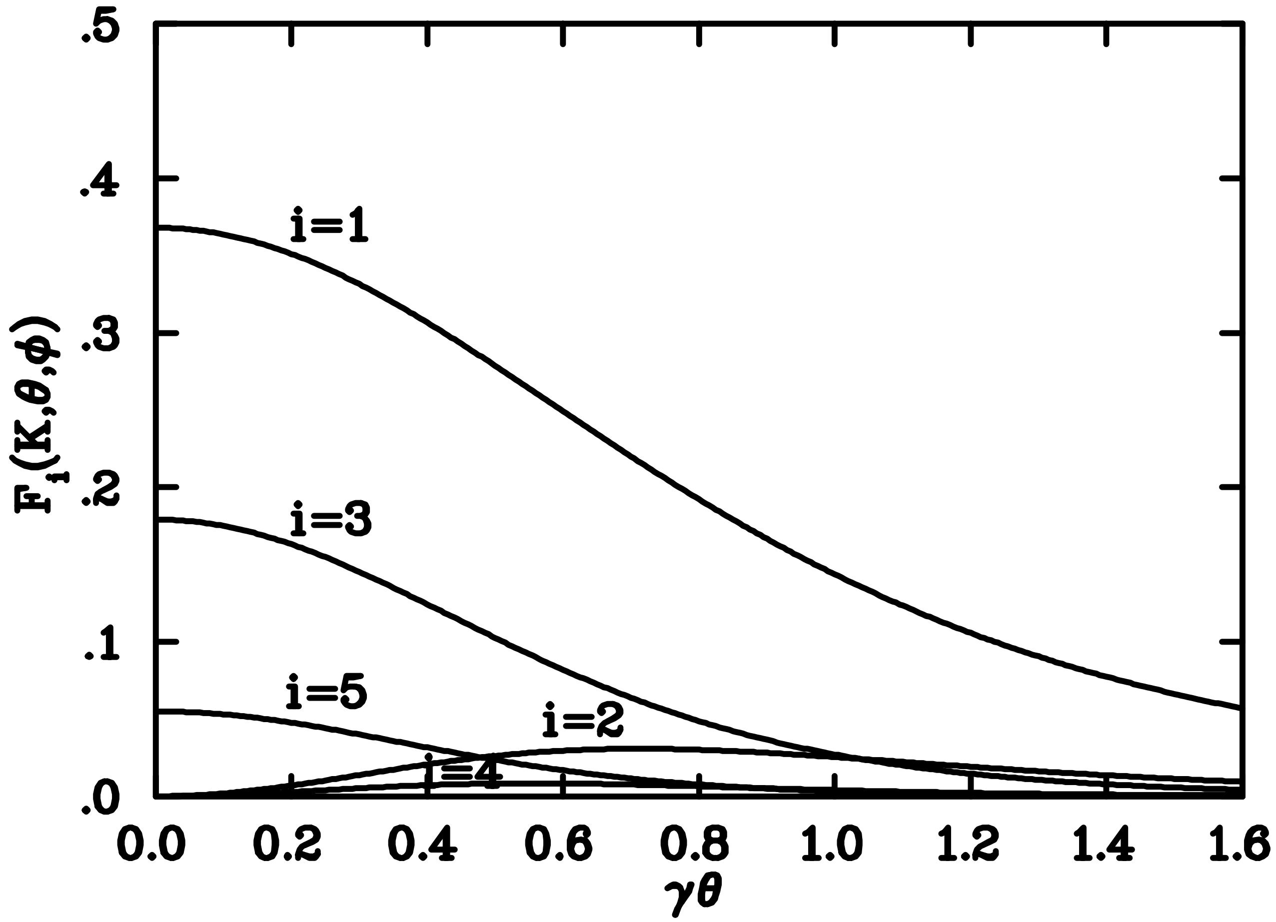
# Radiation integral for a linear undulator (II)

e.g. on axis,

$$\frac{d^2 I}{d\Omega\, d\omega} = \frac{e^2 \gamma^2 N^2}{c} L\left( N \frac{\Delta\omega}{\omega_{res}(\theta)} \right) F_n(K,0,0)$$

$$F_n(K,0,0) = \frac{n^2 K^2}{(1+K^2/2)} \left[ J_{\frac{n+1}{2}}(Z) - J_{\frac{n-1}{2}}(Z) \right]^2 \qquad Z = \frac{nK^2}{4(1+K^2/2)}$$

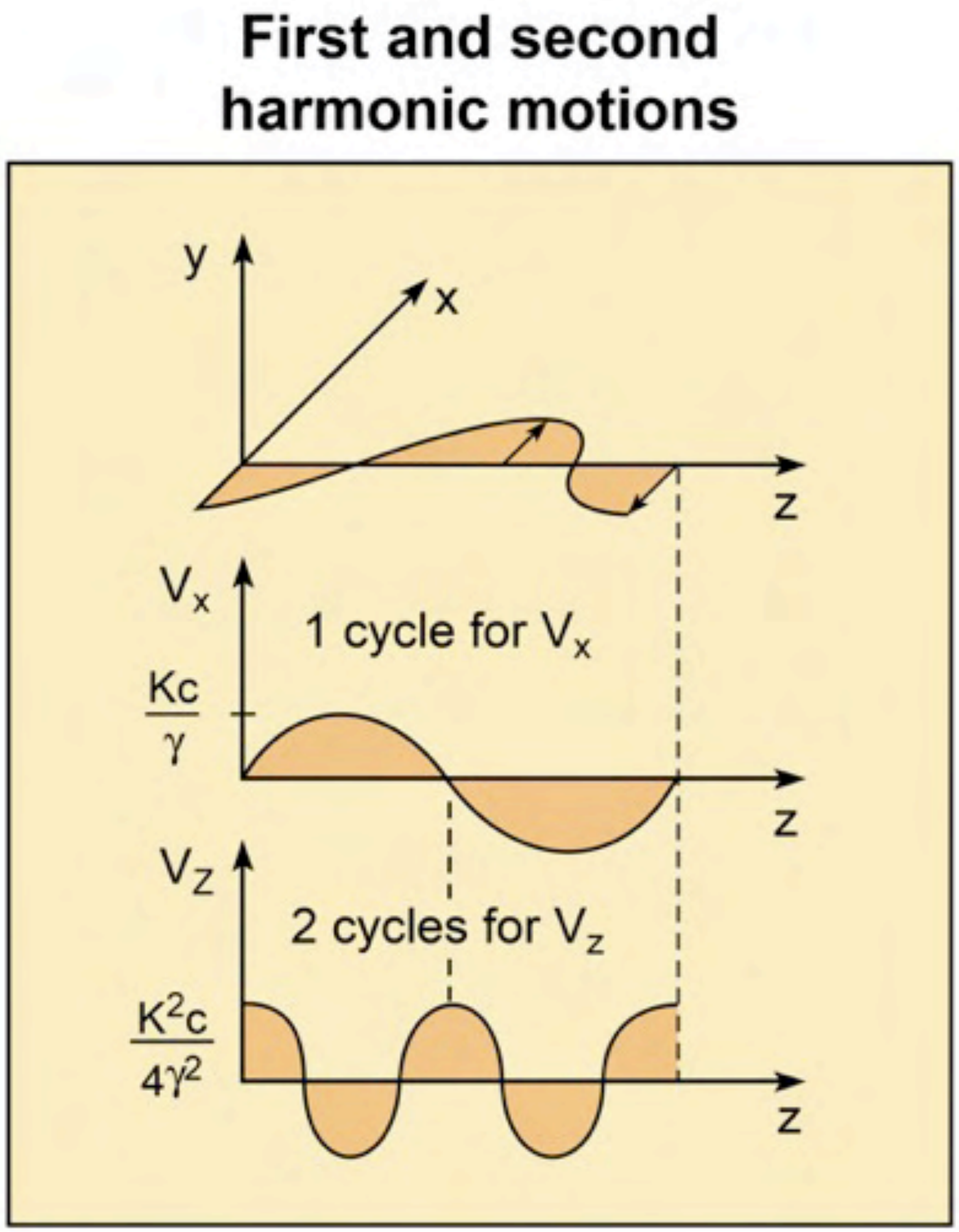


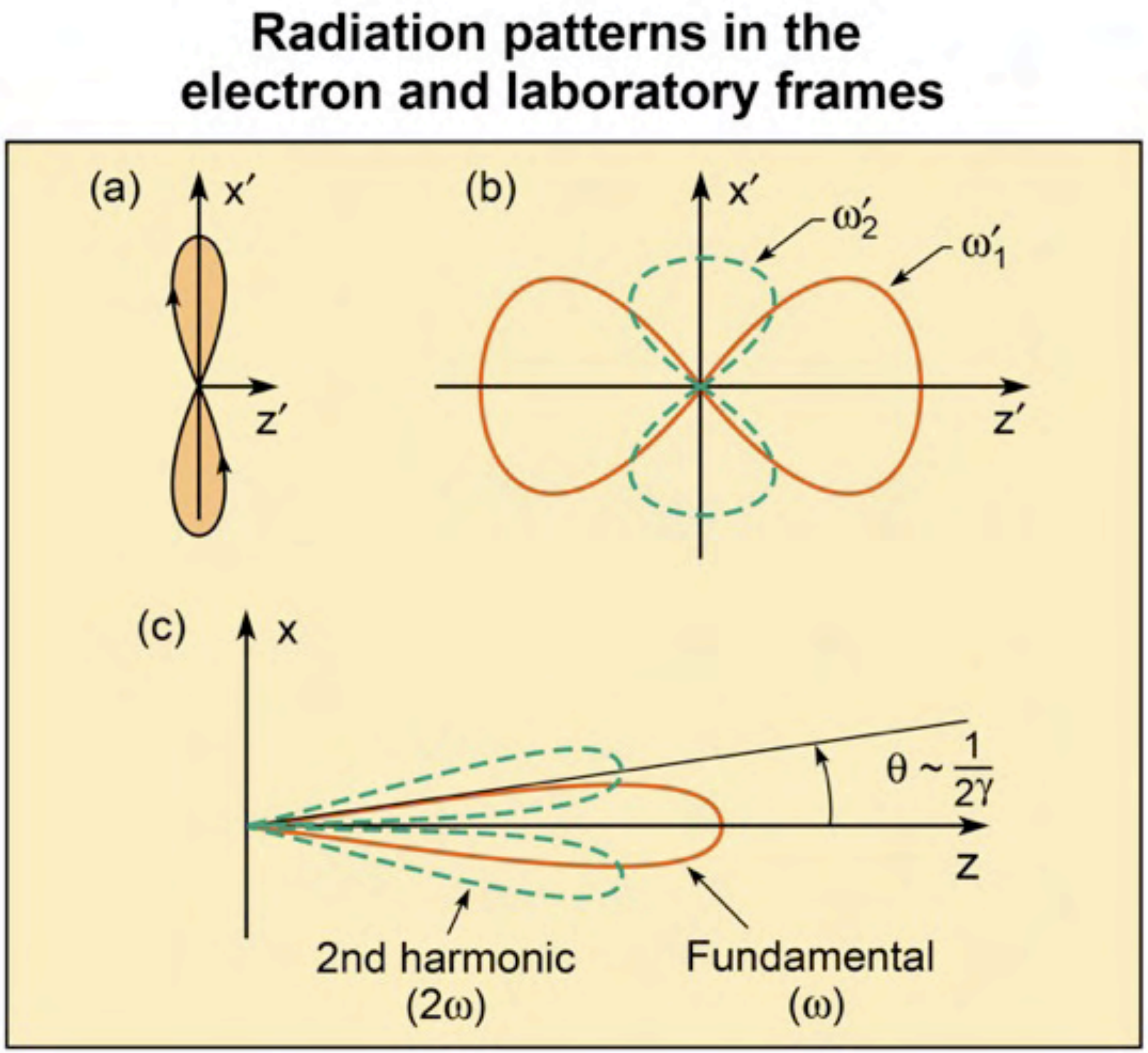




Only odd harmonic are radiated on-axis;

as K increases the harmonic becomes stronger

# Undulator radiation



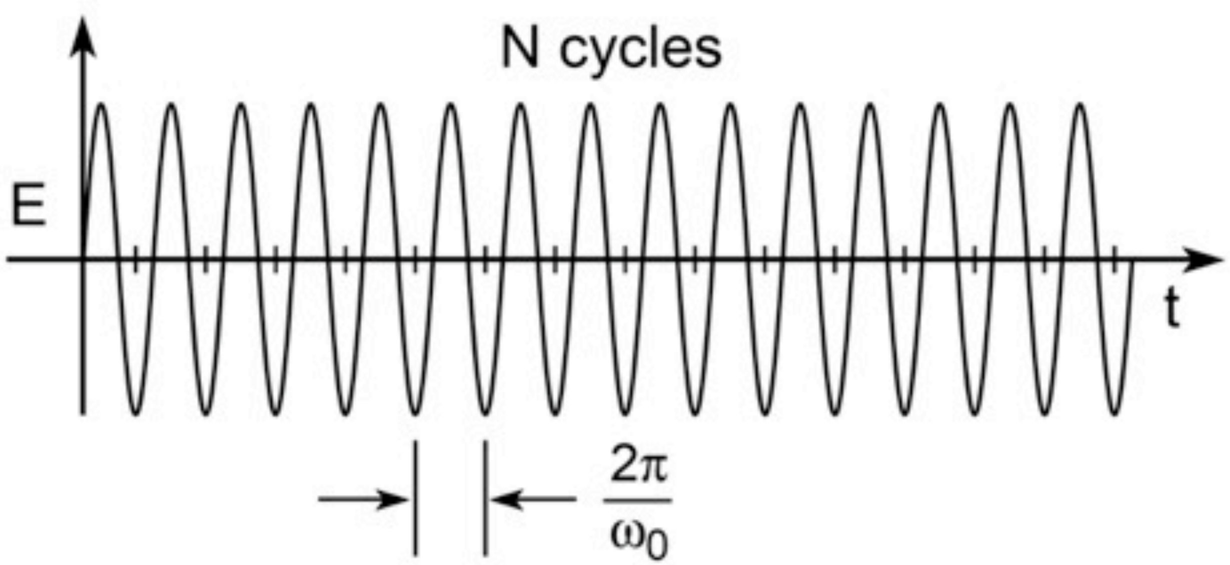

$$\lambda_n = \frac{\lambda_u}{2\gamma^2 n}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right) \tag{5.30}$$

$$\left(\frac{\Delta\lambda}{\lambda}\right)_n = \frac{1}{nN} \tag{5.31}$$

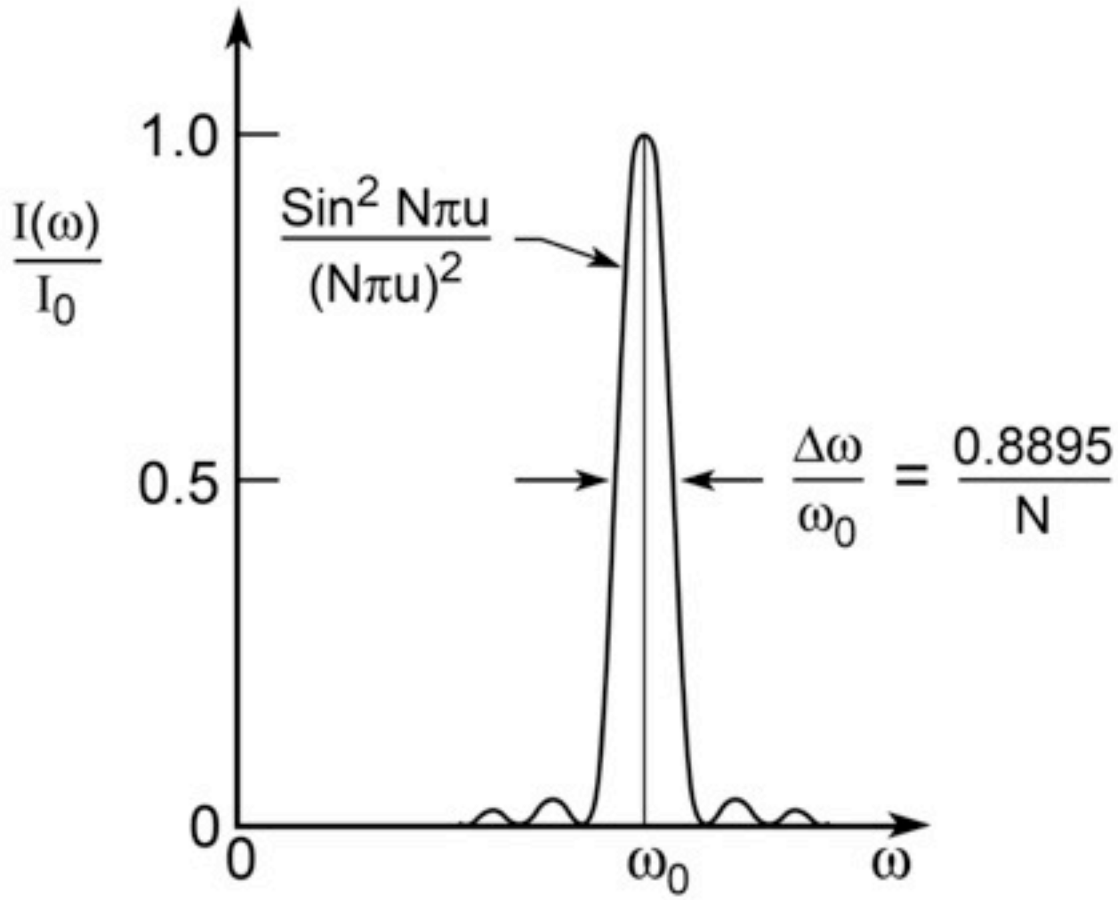
# Undulator radiation

(On-axis radiation, θ = 0)

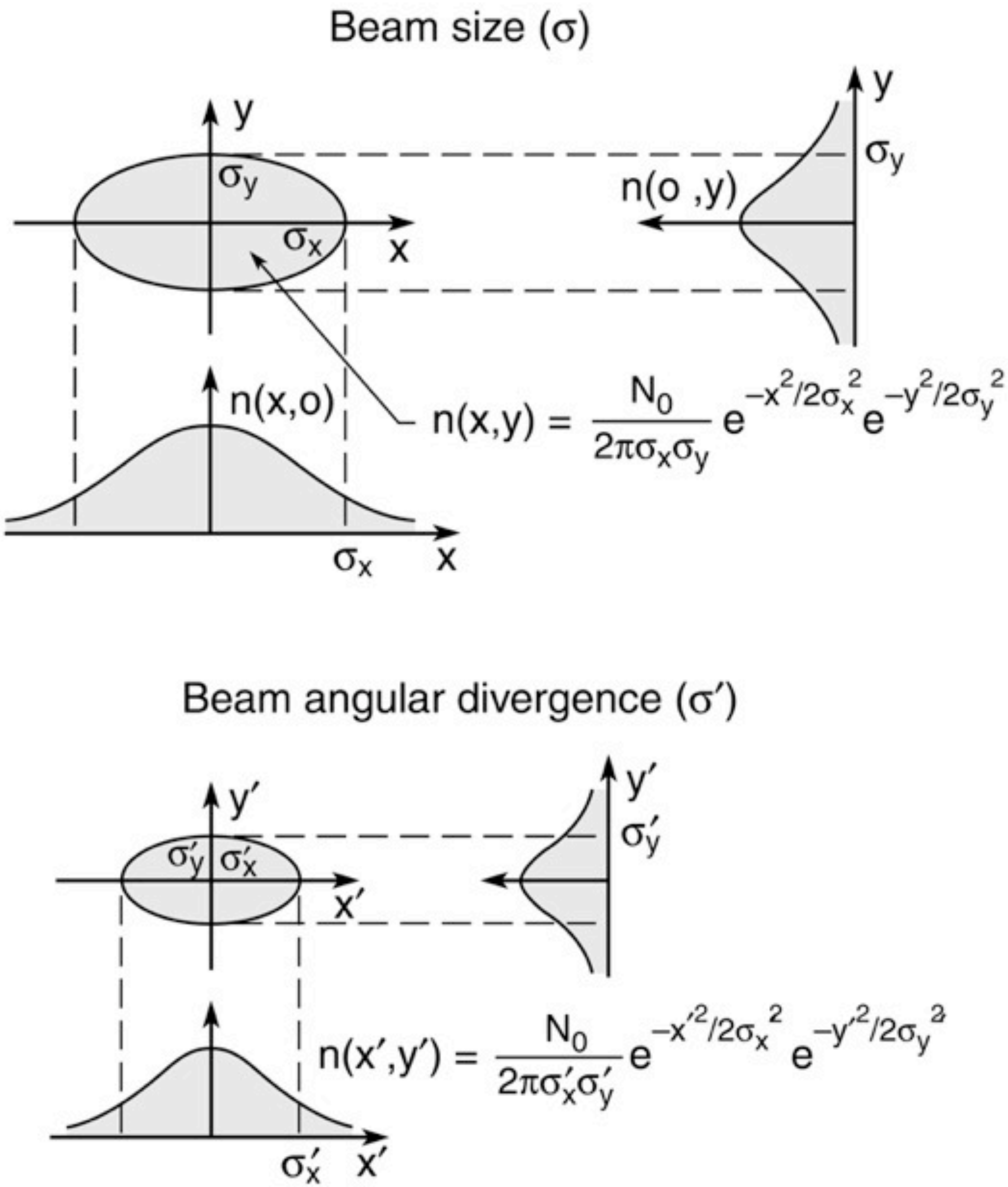
**Radiated Wavetrain**

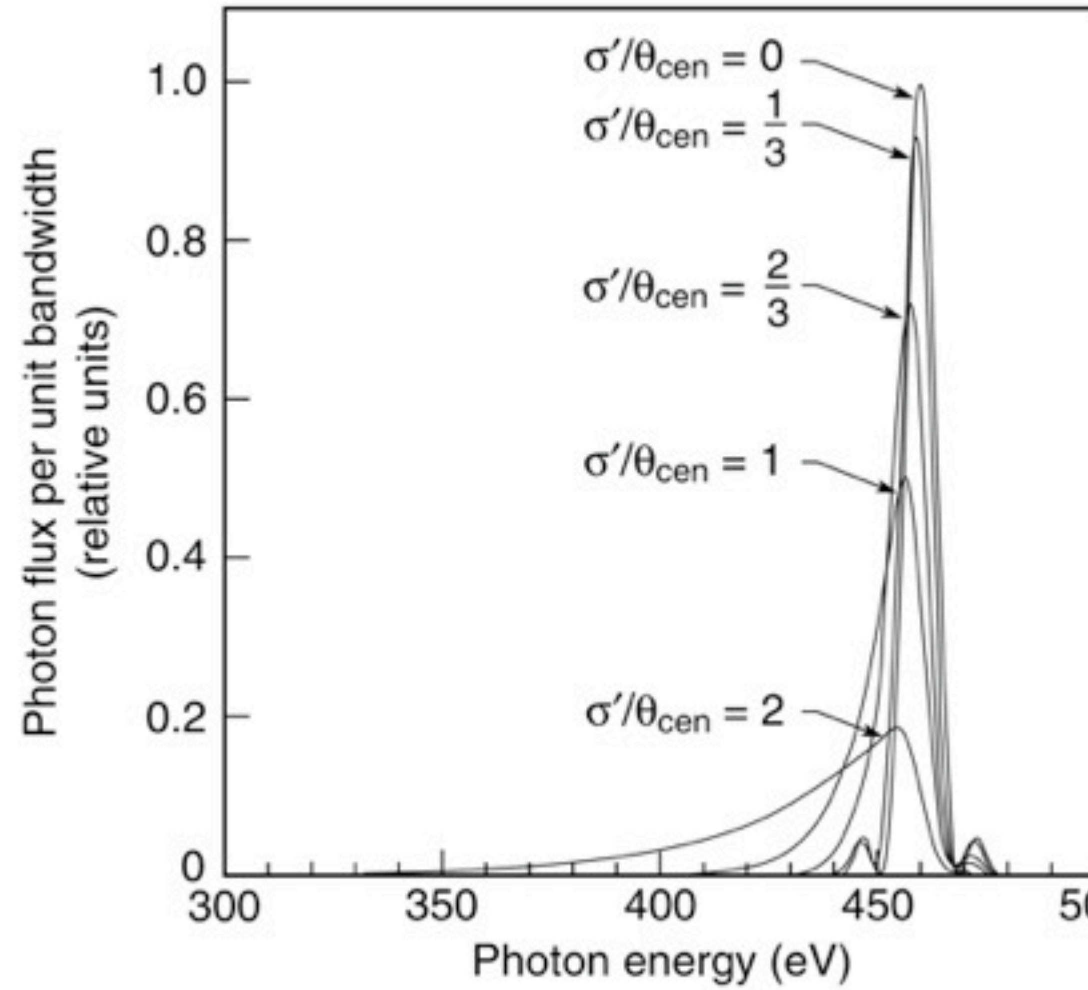



**Spectral Distribution**



# Undulator radiation



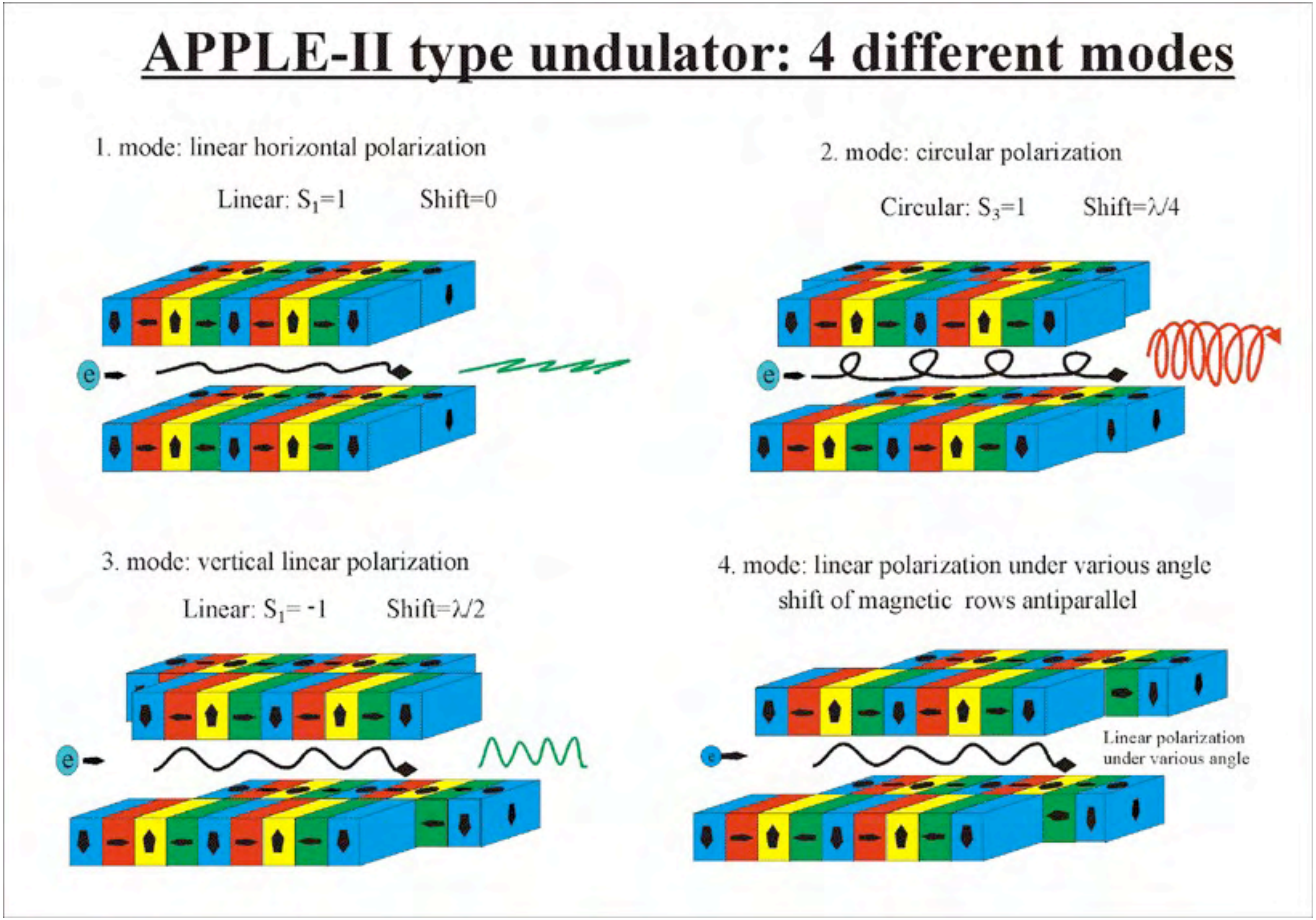

Preserving the spectral line shape of undulator radiation requires

$$\sigma'^2 \ll \theta_{\text{cen}}^2 \tag{5.55b}$$

Define effective, or total central cone half-angles

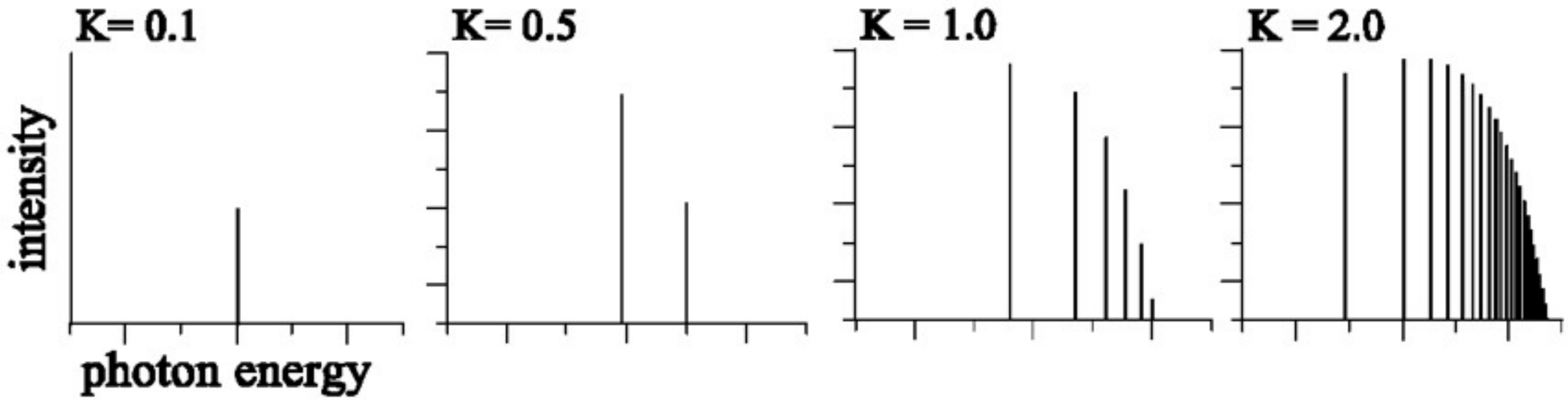
$$\theta_{Tx} = \sqrt{\theta_{\text{cen}}^2 + \sigma_x'^2} \quad \text{and} \quad \theta_{Ty} = \sqrt{\theta_{\text{cen}}^2 + \sigma_y'^2} \tag{5.56}$$

# Undulator radiation



# Undulators and wigglers

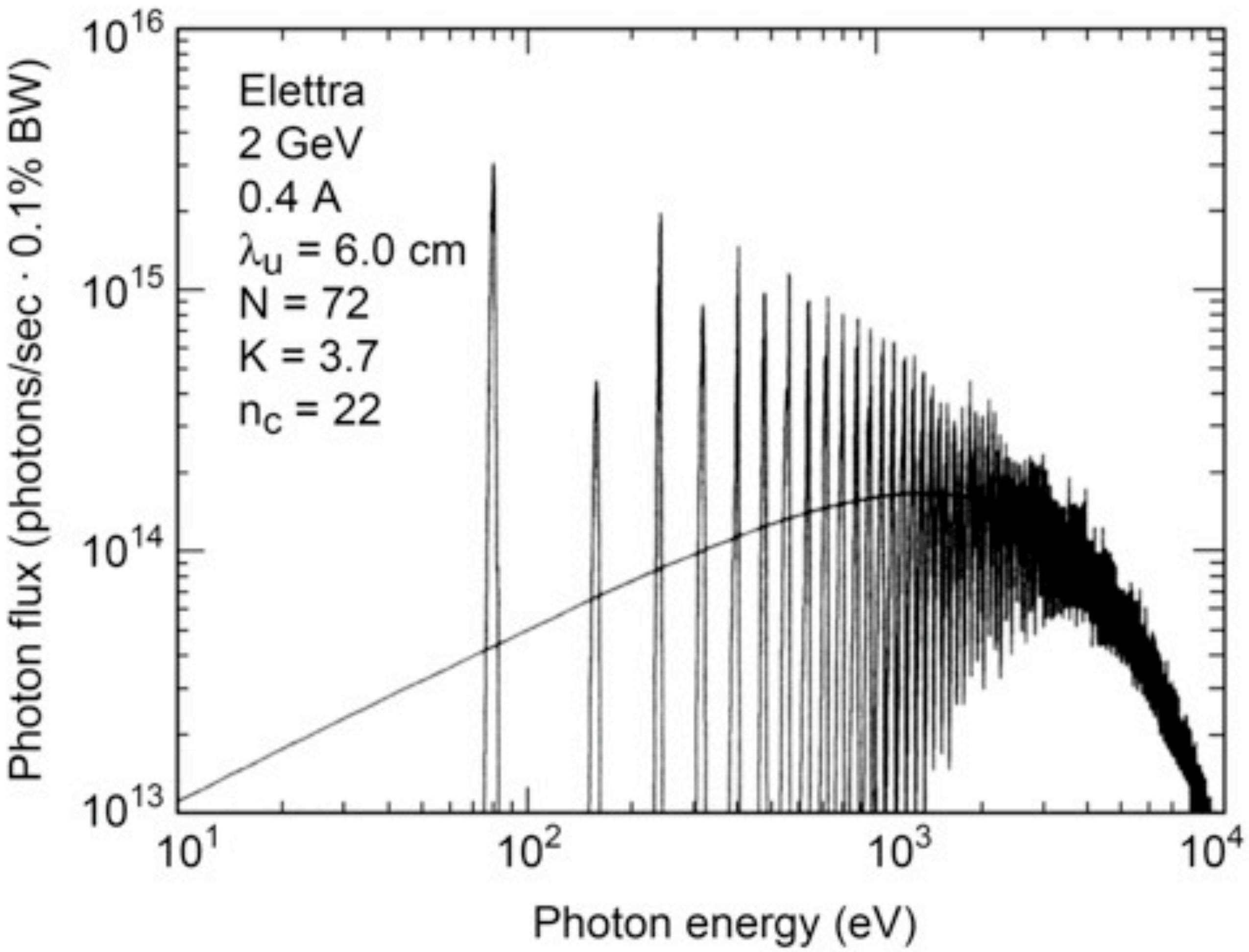
Radiated intensity emitted vs K



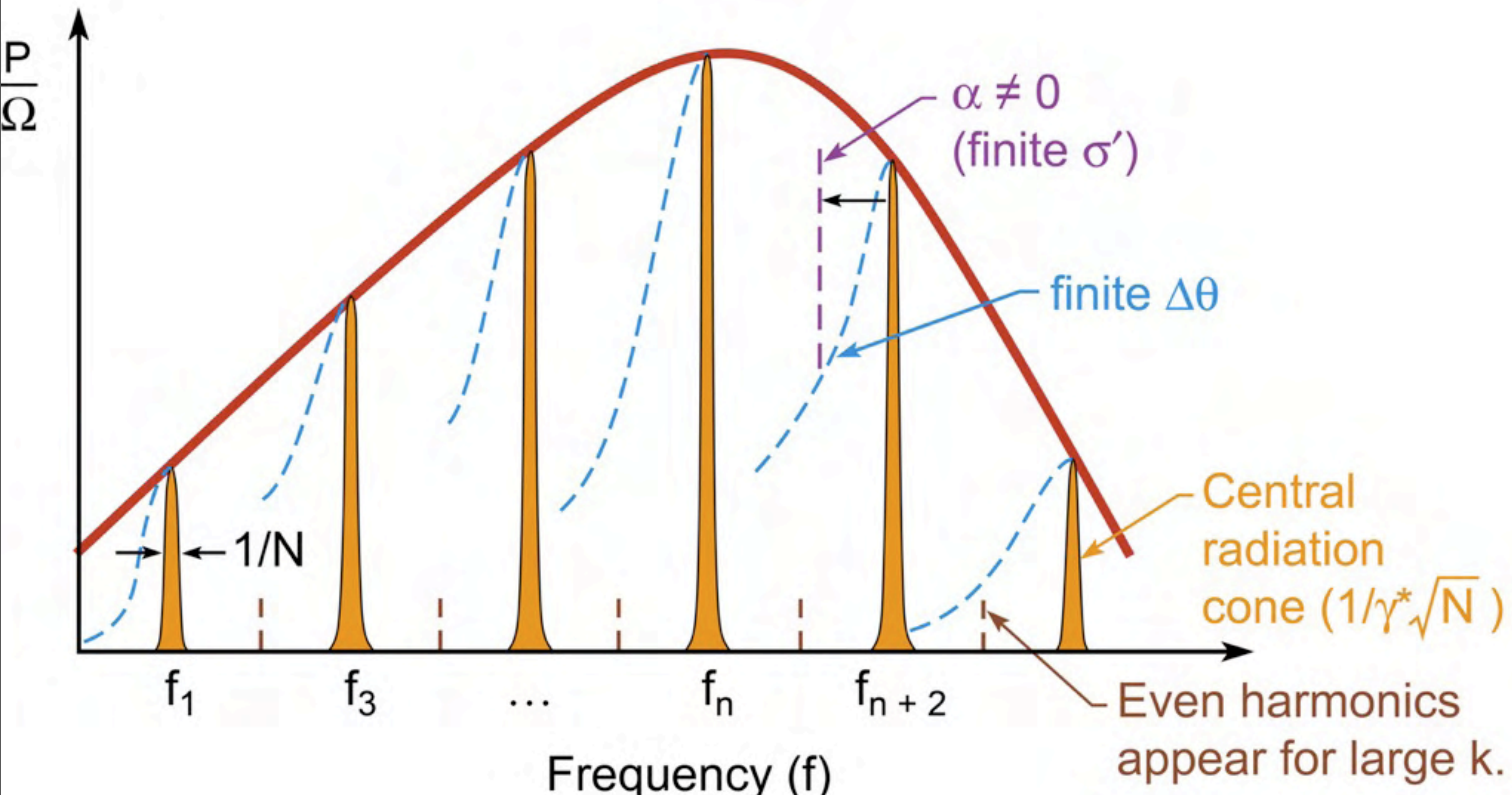

For large K the wiggler spectrum becomes similar to the bending magnet spectrum, $2N_u$ times larger.

Fixed $B_0$, to reach the bending magnet critical wavelength we need:

| | | | | |
|---|---|---|---|---|
| K | 1 | 2 | 10 | 20 |
| n | 1 | 5 | 383 | 3015 |



# Undulators and wigglers



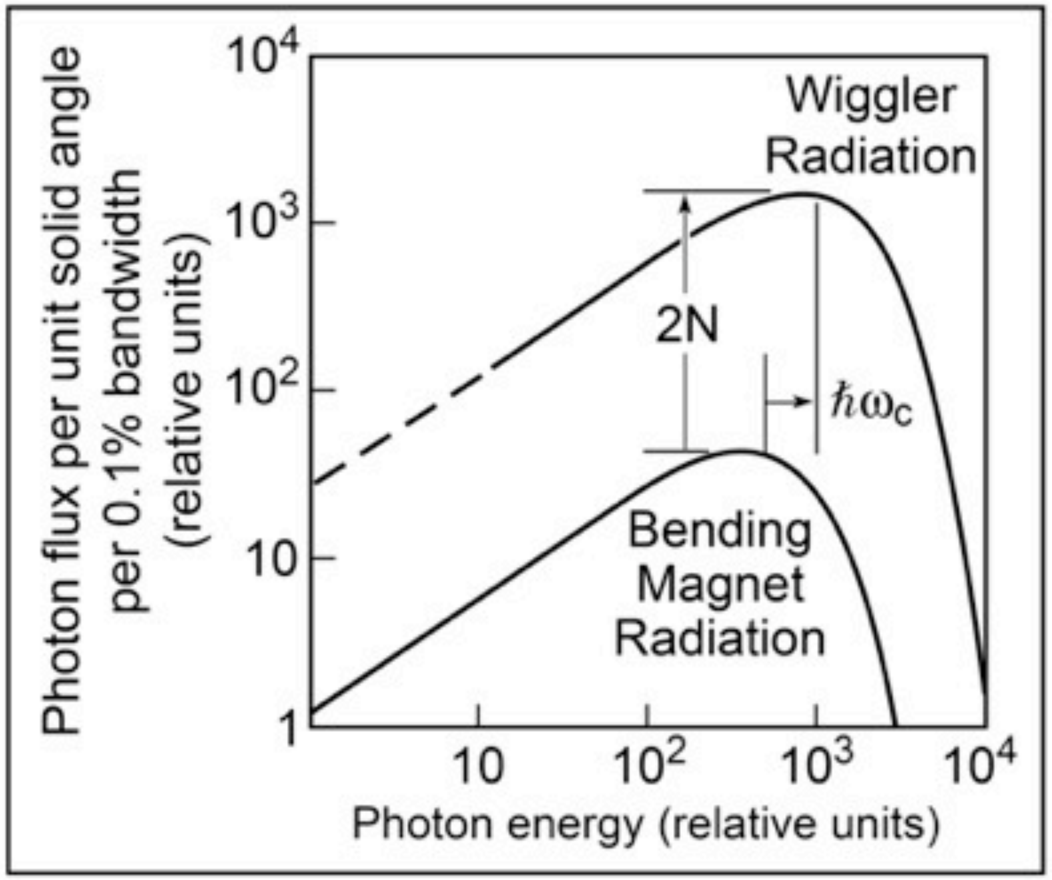
# Undulators and wigglers

At very high K >> 1, the radiated energy appears in very high harmonics, and at rather large horizontal angles $\theta \simeq \pm K/\gamma$ (eq. 5.21). Because the emission angles are large, one tends to use larger collection angles, which tends to spectrally merge nearby harmonics. The result is a continuum at very high photon energies, similar to that of bending magnet radiation, but increased by 2N (the number of magnet pole pieces).

$$E_c = \hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m} \quad ; \quad n_c = \frac{3K}{4}\left(1 + \frac{K^2}{2}\right) \qquad (5.7a \,\&\, 82)$$

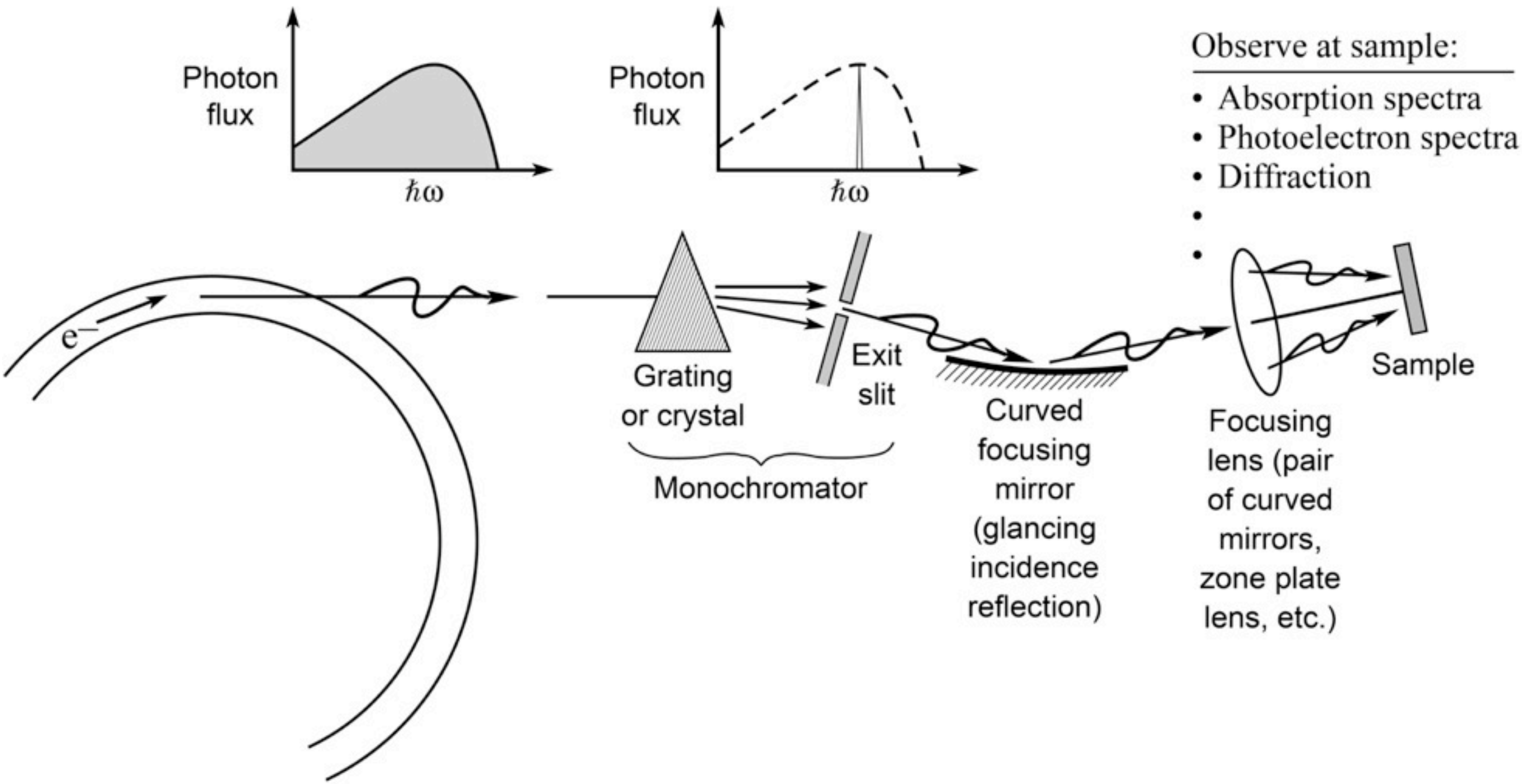
$$\left.\frac{d^2F}{d\theta d\Psi d\omega/\omega}\right|_0 = 2.65 \times 10^{13} N E_e^2(\text{GeV}) I(\text{A}) H_2(E/E_c) \frac{\text{photons/s}}{\text{mrad}^2(0.1\%\text{BW})} \qquad (5.86)$$

$$\frac{d^2F}{d\theta\, d\omega/\omega} = 4.92 \times 10^{13} N E_e(\text{GeV}) I(\text{A}) G_1(E/E_c) \frac{\text{photons/s}}{\text{mrad} \cdot (0.1\%\text{BW})} \qquad (5.87)$$



| Facility | ALS | ELETTRA | Australian Synchrotron | APS |
|---|---|---|---|---|
| Electron energy | 1.90 GeV | 2.0 GeV | 3.0 GeV | 7.00 GeV |
| $\gamma$ | 3720 | 3910 | 5871 | 13,700 |
| Current (mA) | 400 | 300 | 200 | 100 |
| Circumference (m) | 197 | 259 | 216 | 1100 |
| RF frequency (MHz) | 500 | 500 | 500 | 352 |
| Pulse duration (FWHM) (ps) | 35-70 | 37 | ~100 | 100 |
| *Bending Magnet Radiation:* | | | | |
| Bending magnet field (T) | 1.27 | 1.2 | 1.31 | 0.599 |
| Critical photon energy (keV) | 3.05 | 3.2 | 7.84 | 19.5 |
| Critical photon wavelength | 0.407 nm | 0.39 nm | 1.58 Å | 0.636 Å |
| Bending magnet sources | 24 | 12 | 28 | 35 |
| *Undulator Radiation:* | | | | |
| Number of straight sections | 12 | 12 | 14 | 40 |
| Undulator period (typical) (cm) | 5.00 | 5.6 | 22.0 | 3.30 |
| Number of periods | 89 | 81 | 80 | 72 |
| Photon energy ($K = 1$, $n = 1$) | 457 eV | 452 eV | 2.59 keV | 9.40 keV |
| Photon wavelength ($K = 1$, $n = 1$) | 2.71 nm | 2.74 nm | 0.478 nm | 1.32 Å |
| Tuning range ($n = 1$) | 230-620 eV | 2.0-6.7 nm | 0.319-0.835 nm | 3.5-12 keV |
| Tuning range ($n = 3$) | 690-1800 eV | 0.68-2.2 nm | 0.106-0.278 nm | 10-38 keV |
| Central cone half-angle ($K = 1$) | 35 μrad | 35 μrad | 23 μrad | 11 μrad |
| Power in central cone ($K = 1$, $n = 1$) (W) | 2.3 | 1.7 | 6.6 | 12 |
| Flux in central cone (photons/s) | $3.1 \times 10^{16}$ | $2.3 \times 10^{16}$ | $1.6 \times 10^{16}$ | $7.9 \times 10^{15}$ |
| $\sigma_x$, $\sigma_y$ (μm) | 260, 16 | 255, 23 | 320, 16 | 320, 50 |
| $\sigma'_x$, $\sigma'_y$ (μrad) | 23, 3.9 | 31, 9 | 34, 6 | 23, 7 |
| Brightness ($K = 1$, $n = 1$)[a] [(photons/s)/mm$^2$ · mrad$^2$ · (0.1%BW)] | $2.3 \times 10^{19}$ | $9.9 \times 10^{18}$ | $1.3 \times 10^{19}$ | $5.9 \times 10^{18}$ |
| Total power ($K = 1$, all $n$, all $\theta$) (W) | 83 | 126 | 476 | 350 |
| Other undulator periods (cm) | 3.65, 8.00, 10.0 | 8.0, 12.5 | 6.8, 18.3 | 2.70, 5.50, 12.8 |
| *Wiggler Radiation:* | | | | |
| Wiggler period (typical) (cm) | 16.0 | 14.0 | 6.1 | 8.5 |
| Number of periods | 19 | 30 | 30 | 28 |
| Magnetic field (maximum) (T) | 2.1 | 1.5 | 1.9 | 1.0 |
| $K$ (maximum) | 32 | 19.6 | 12 | 7.9 |
| Critical photon energy (keV) | 5.1 | 4.0 | 11.4 keV | 33 |
| Critical photon wavelength | 0.24 nm | 0.31 nm | 0.11 nm | 0.38 Å |
| Total power (max. $K$) (kW) | 13 | 7.2 | 9.3 | 7.4 |

[a]Using Eq. (5.65). See comments following Eq. (5.64) for the case where $\sigma'_{x,y} \simeq \theta_{cen}$.


Photon
flux
ħω
Photon
flux
ħω
Observe at sample:
• Absorption spectra
• Photoelectron spectra
• Diffraction
•
•
e⁻
Grating
or crystal
Exit
slit
Monochromator
Curved
focusing
mirror
(glancing
incidence
reflection)
Focusing
lens (pair
of curved
mirrors,
zone plate
lens, etc.)
Sample