Abstract

Quantum and free-electron lasers (FELs) are based on distributed interactions between electromagnetic radiation and gain media. In an amplifier configuration, a forward wave is amplified while propagating in a polarized medium. Formulating a coupled mode theory for excitation of both forward and backward waves, we identify conditions for phase matching, leading to efficient excitation of backward wave without any mechanism of feedback or resonator assembly. The excitations of incident and reflected waves are described by a set of coupled differential equations expressed in the frequency domain. The induced polarization is given in terms of an electronic susceptibility tensor. In quantum lasers the interaction is described by two first order differential equations, while in high-gain free-electron lasers, the differential equations are of the third order each. Analytical solutions of reflectance and transmittance for both quantum lasers and FELs are presented. It is found that when the solutions become infinite, the device operates as an oscillator, producing radiation at the output with no field at its input, entirely without any localized or distributed feedback.

INTRODUCTION

Conventional (quantum) lasers, microwave tubes and free-electron lasers (FELs) are based on distributed interactions between electromagnetic radiation and gain media. When such devices are operating in an amplifier configuration, a forward wave is amplified while propagating in a polarized medium, in a stimulated emission process [1]. In an oscillator configuration a resonator [2]-[4] or a distributed feedback [5] are employed to circulate the radiation, which is excited and amplified by the gain medium. If the single-pass gain is higher than the total losses, the radiation intensity inside the cavity increases and becomes more coherent. After several round trips, the radiation is built up until arriving at the nonlinear regime and saturation.

In this paper we suggest a mechanism of generation of laser oscillations, without any feedback means. It is shown that under conditions of phase-matching, both forward and backward waves can be excited in a distributed gain medium as illustrated schematically in Figure 1. The excitation of incident and reflected waves is described by a set of two differential equations coupled by the induced polarization of the gain media. The coupling coefficient is given in terms of the electronic susceptibility tensor.

Two cases are discussed: In quantum lasers, which are characterized by isotropic, homogeneous gain media, the interaction is described by two first order differential equations. In high-gain free-electron lasers, where the susceptibility is space dependent, the set includes two differential equations of the third order each. The coupled equations sets are solved analytically for both cases. Oscillation conditions are identified from the derived reflectance and transmittance coefficients.

EXCITATION OF FORWARD AND BACKWARD MODES

The total electromagnetic field is given by the time harmonic wave vector:

$$ E(r, t) = \Re \{ \tilde{E}(r) e^{-j\omega t} \} $$

where \( \tilde{E}(r) \) is the phasor of the wave oscillating at an angular frequency \( \omega \). The vector \( r \) stands for the \((x, y, z)\) coordinates, where \((x, y)\) are the transverse coordinates and
$z$ is the axis of propagation. In the case of excitation of forward and backward modes, the phasor can be written as the sum [6]:

$$\mathbf{E}(r) = [C_+(z)e^{jkz} + C_-(z)e^{-jkz}] \mathbf{\tilde{E}}(x,y)$$  \hspace{1cm} (2)

$C_+(z)$ and $C_-(z)$ are scalar amplitudes of forward and backward modes respectively, with profile $\mathbf{\tilde{E}}(x,y)$ and axial wavenumber $k_z$. The evolution of the amplitudes of the excited modes is described by a set of two coupled differential equations:

$$\frac{d}{dz}C_{\pm}(z) = \pm\frac{1}{2N}e^{\mp jkz} \int \int \mathbf{\tilde{J}}(r) \cdot \mathbf{\tilde{E}}^*(x,y) \, dx \, dy$$ \hspace{1cm} (3)

The normalization of the mode amplitudes is made via the complex Poynting vector power:

$$\mathcal{N} = \int \int [\mathbf{\tilde{E}}_L(x,y) \times \mathbf{\tilde{H}}_L^*(x,y)] \cdot \mathbf{\hat{z}} \, dx \, dy$$  \hspace{1cm} (4)

The total power carried by the forward and backward (propagating) modes is:

$$P(z) = \frac{1}{2} \Re \int \int [\mathbf{\tilde{E}}(r) \times \mathbf{\tilde{H}}^*(r)] \cdot \mathbf{\hat{z}} \, dx \, dy$$

$$= \frac{1}{2} \left[|C_+(z)|^2 - |C_-(z)|^2 \right] \cdot \Re \{\mathcal{N}\}$$ \hspace{1cm} (5)

When the interaction takes place in a polarized gain medium, the driving current density $\mathbf{J}(r)$ is given in terms of the induced polarization (dipole moment per unit volume) $\mathbf{P}(r)$. In the time domain, the current density is the time derivative of the induced polarization. Thus, the phasor representation of the driving current density is given by:

$$\mathbf{\tilde{J}}(r) = -j\omega\mathbf{\tilde{P}}(r) = -j\omega\varepsilon_0 \chi(r,\omega) \cdot \mathbf{\tilde{E}}(r)$$ \hspace{1cm} (6)

where $\chi(r,\omega)$ is the electronic susceptibility tensor at the frequency $\omega$ (in a homogeneous isotropic medium it is a scalar). Using (6) in (3) results in:

$$\frac{d}{dz}C_{\pm}(z) =$$

$$\pm j\frac{\omega\varepsilon_0}{2N}e^{\mp jkz} \int \int \mathbf{\tilde{E}}(r) \cdot \chi(r,\omega) \cdot \mathbf{\tilde{E}}^*(x,y) \, dx \, dy$$ \hspace{1cm} (7)

Substitution of the field expansion (2) in the excitation equations (7), the mode amplitudes $C_{\pm}(z)$ are described by a set of two coupled differential equations, that can be presented in a matrix form:

$$\frac{d}{dz} \begin{bmatrix} C_+(z) \\ C_-(z) \end{bmatrix} =$$

$$\begin{bmatrix} +\kappa(z) & +\kappa(z)e^{-jkz} \\ -\kappa(z)e^{jkz} & -\kappa(z) \end{bmatrix} \begin{bmatrix} C_+(z) \\ C_-(z) \end{bmatrix}$$ \hspace{1cm} (8)

The coupling parameter:

$$\kappa(z,\omega) \equiv j\frac{\omega\varepsilon_0}{2N} \int \int \mathbf{\tilde{E}}(x,y) \cdot \chi(r,\omega) \cdot \mathbf{\tilde{E}}^*(x,y) \, dx \, dy$$ \hspace{1cm} (9)

is in general a complex, space-frequency dependent quantity.

### QUANTUM LASER

We relate first to gain media, where the electronic susceptibility does not change along the axis of propagation $z$. This situation occurs in quantum lasers, where the atomic susceptibility of the gain medium is uniform [1]. In that case the coupling parameter is not yet space ($z$) dependent and can be presented in the form $\kappa(\omega) = \gamma(\omega) + j\beta(\omega)$, where $\gamma(\omega)$ is the field gain factor. Consequently, the set (8) can be written as two coupled first order linear differential equations:

$$\frac{d}{dz} \begin{bmatrix} C_+(z) \\ C_-(z) \end{bmatrix} =$$

$$\begin{bmatrix} +\kappa & +\kappa e^{-jkz} \\ -\kappa e^{jkz} & -\kappa \end{bmatrix} \begin{bmatrix} C_+(z) \\ C_-(z) \end{bmatrix}$$ \hspace{1cm} (10)

Analytical solution of the coupled set (10) for a given forward mode amplitude $C_+(0)$ at the input $z = 0$, while the backward mode amplitude at the exit of the interaction region ($z = L$) is $C_-(L) = 0$, leads to the solution of incident and reflected wave amplitudes:

$$\frac{C_+(z)}{C_+(0)} =$$

$$\frac{(\kappa + jk_z) \sinh [S(L-z)] - S \cosh [S(L-z)]}{(\kappa + jk_z) \sinh (SL) - S \cosh (SL)} e^{-jk_zz}$$ \hspace{1cm} (11)

$$\frac{C_-(z)}{C_+(0)} =$$

$$\frac{-\kappa \sinh [S(L-z)]}{(\kappa + jk_z) \sinh (SL) - S \cosh (SL)} e^{jk_zz}$$

where $S \equiv \sqrt{(\kappa + jk_z)^2 - \kappa^2}$ is a complex parameter. The evolution of incident and reflected wave amplitudes along the gain medium are shown in Fig. 2. It is assumed that the interaction takes place in the vicinity of the resonance frequency, where $\kappa(\omega_0)$ is real.

Figure 2: The evolution of (a) incident and (b) reflected wave amplitudes along the gain medium.
The transmission gain is defined by:
\[
\frac{C_+(L)}{C_+(0)} = \frac{-SL}{(\kappa + jk_z)L \sinh(SL) - SL \cosh(SL)} e^{-jk_z L} 
\]
(12)
Respectively, the reflection gain is:
\[
\frac{C_-(0)}{C_+(0)} = \frac{-\kappa L \sinh(SL)}{(\kappa + jk_z)L \sinh(SL) - SL \cosh(SL)} 
\]
(13)
Contour plots of the transmission and reflection power gain in the \((k_z L, \kappa L)\) plane are shown in Figure 3. An infinite gain limit is obtained when an infinite gain is obtained.

### ACKNOWLEDGMENTS

The research was supported by the Israel Science Foundation.

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Figure 3: (a) Transmission and (b) reflection contours in the \((k_z L, \kappa L)\) plane for atomic laser.
Equation (19) can be readily solved and the solutions are:

\[ \gamma_1 = -1 + ba - \frac{a^2}{b} \]

\[ \gamma_2 = \frac{1}{2} \left[ -\gamma_1 (1 + \sqrt{3}j) - 3 - \sqrt{3}j + 2\sqrt{3}jab \right] \]

\[ \gamma_3 = \frac{1}{2} \left[ -\gamma_1 (1 - \sqrt{3}j) - 3 + \sqrt{3}j - 2\sqrt{3}jab \right] \]

where:

\[ a = (-jw)^\frac{3}{4} \quad b = (2 + \sqrt{4 - jw})^\frac{3}{4} \]

We see that:

\[ \gamma_3 (w) = \gamma_1^\frac{3}{2} (-w) \]

Hence one can deduce the behavior of the roots for negative \( w \) values from their behavior for positive values.

In the case \( w \to 0 \) we obtain \( \lim_{w \to 0} \gamma_1 = -1 \) for \( i \in [1, 2, 3] \). This is obviously a degenerate case in which the solution is not an exponential type. Inserting the condition \( w \to 0 \) into equation (15) one is left with the trivial equation \( \frac{d^2}{dz^2} \left[ C_+ (z) \right] = 0 \) with the solution:

\[ C_+ (z) = \frac{1}{2} z^2 c_{(2)}^+ + z c_{(1)}^+ + c_{(0)}^+ \]

\[ C_- (z) = \frac{1}{2} z^2 c_{(2)}^- + z c_{(1)}^- + c_{(0)}^- \]

In the opposite limit in which \( w \to \infty \) we obtain the following results:

\[ \lim_{w \to \infty} \gamma_1 = \frac{1}{3} \]

\[ \lim_{w \to \infty} \gamma_2 = \sqrt{3} e^{\frac{2}{3} \pi j w^\frac{3}{4}} \]

\[ \lim_{w \to \infty} \gamma_3 = \sqrt{3} e^{\frac{2}{3} \pi j w^\frac{3}{4}} \]

Since none of those are negative real numbers this means that we have three growing exponents and three decaying exponents in the case of large \( w \) those are:

\[ \lim_{w \to \infty} \lambda_1 = \frac{1}{\sqrt{3}} \]

\[ \lim_{w \to \infty} \lambda_2 = -\frac{1}{\sqrt{3}} \]

\[ \lim_{w \to \infty} \lambda_3 \cong 3^{\frac{1}{4}} (-0.383 + 0.924j) w^{\frac{3}{4}} \]

\[ \lim_{w \to \infty} \lambda_4 \cong 3^{\frac{1}{4}} (0.383 - 0.924j) w^{\frac{3}{4}} \]

\[ \lim_{w \to \infty} \lambda_5 \cong 3^{\frac{1}{4}} (0.924 + 0.383j) w^{\frac{3}{4}} \]

\[ \lim_{w \to \infty} \lambda_6 \cong 3^{\frac{1}{4}} (-0.924 - 0.383j) w^{\frac{3}{4}} \]

For large \( w \), \( \lambda_5 \) is clearly the most dominant exponent. Although \( \lambda_1 \) & \( \lambda_2 \) asymptotically approach a finite number, the other eigenvalues continue to grow without limit.