FEL beam characterization by measurement of wavefront and mutual coherence function

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Dept.
“Optics / Short Wavelengths”

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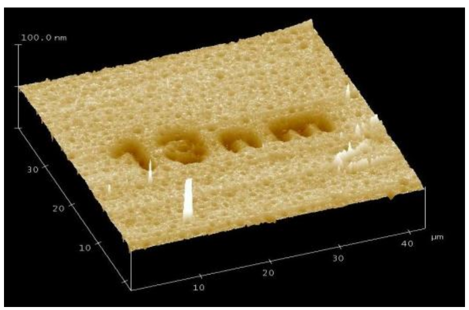
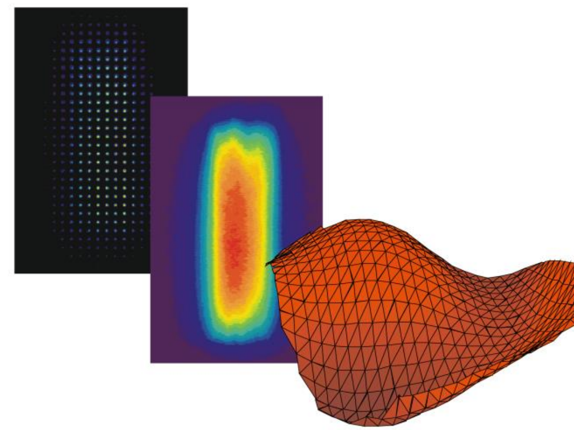
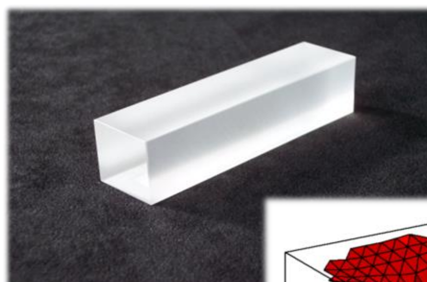
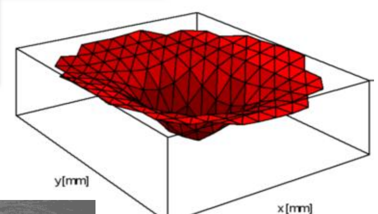


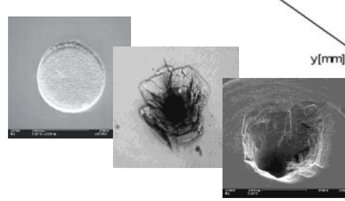
Optics test (351…193 nm)

- *(Long term) degradation (10^9 pulses)*
- *Non-linear processes*
- *LIDT*
- *Absorption / Scatter losses*
- *Wavefront deformation*









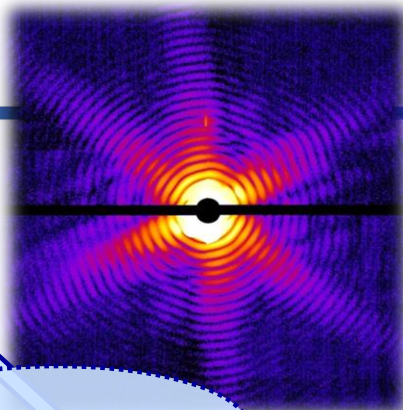
EUV/XUV technology

- *Source & Optics*
- *Metrology*
- *Material interaction*

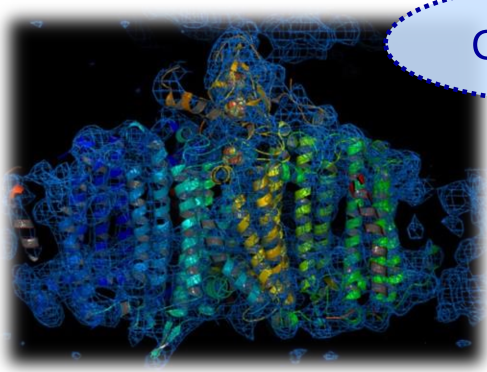
Beam characterization

- *Wavefront*
- *Coherence*
- *M²*

Motivation



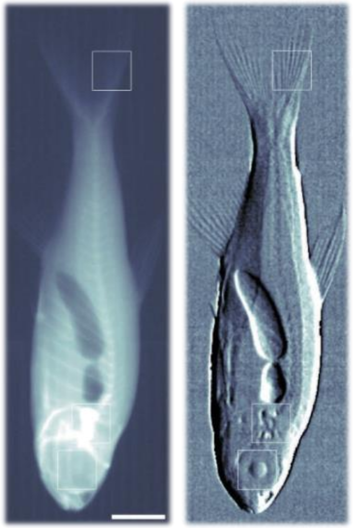
Coherent Diffractive Imaging



Coherence

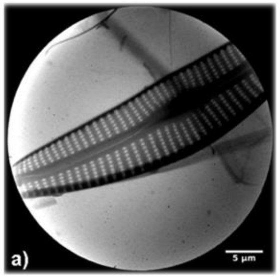
Talbot interferometry

Wavefront

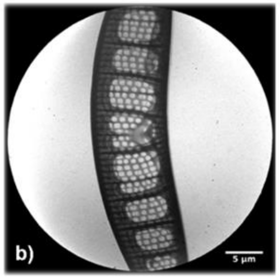


Soft X-ray source (FEL, HHG, LPP)

Focusability



X-ray microscopy



Principle of Hartmann sensor

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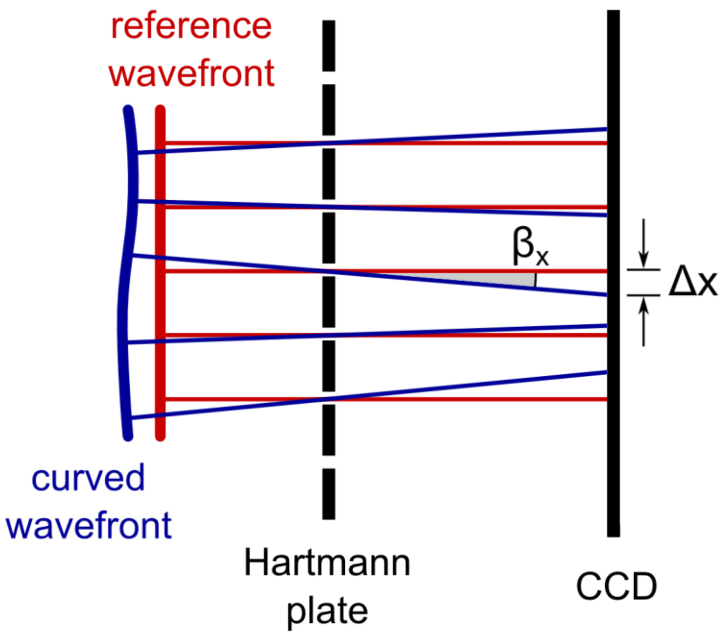
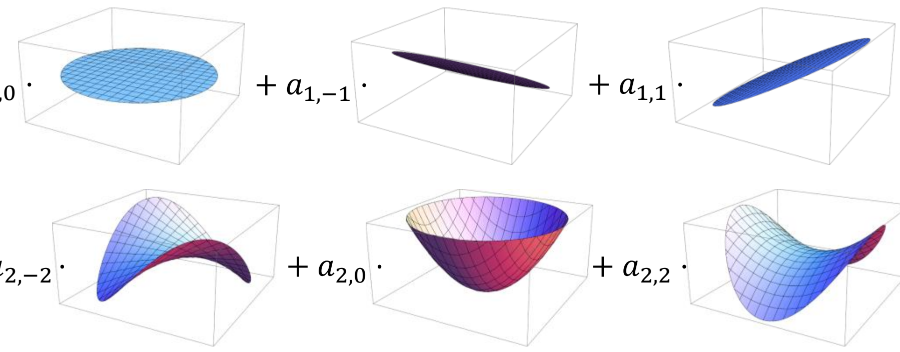


$$\nabla w(x,y) = \begin{pmatrix} \beta_x \\ \beta_y \end{pmatrix}$$

Zernike polynomials

$$w(\rho,\phi) = a_{0,0} + a_{1,-1}\cdot\rho\sin\phi + a_{1,1}\cdot\rho\cos\phi$$
$$+ a_{2,-2}\cdot(2\rho^2-1)\sin\phi + a_{2,0}\cdot(2\rho^2-1) + a_{2,2}\cdot(2\rho^2-1)\cos\phi$$
$$+\cdots$$



$$= a_{0,0}\cdot[\text{plot}] + a_{1,-1}\cdot[\text{plot}] + a_{1,1}\cdot[\text{plot}]$$
$$+ a_{2,-2}\cdot[\text{plot}] + a_{2,0}\cdot[\text{plot}] + a_{2,2}\cdot[\text{plot}]$$
$$+\cdots$$

Beam characterization of FLASH

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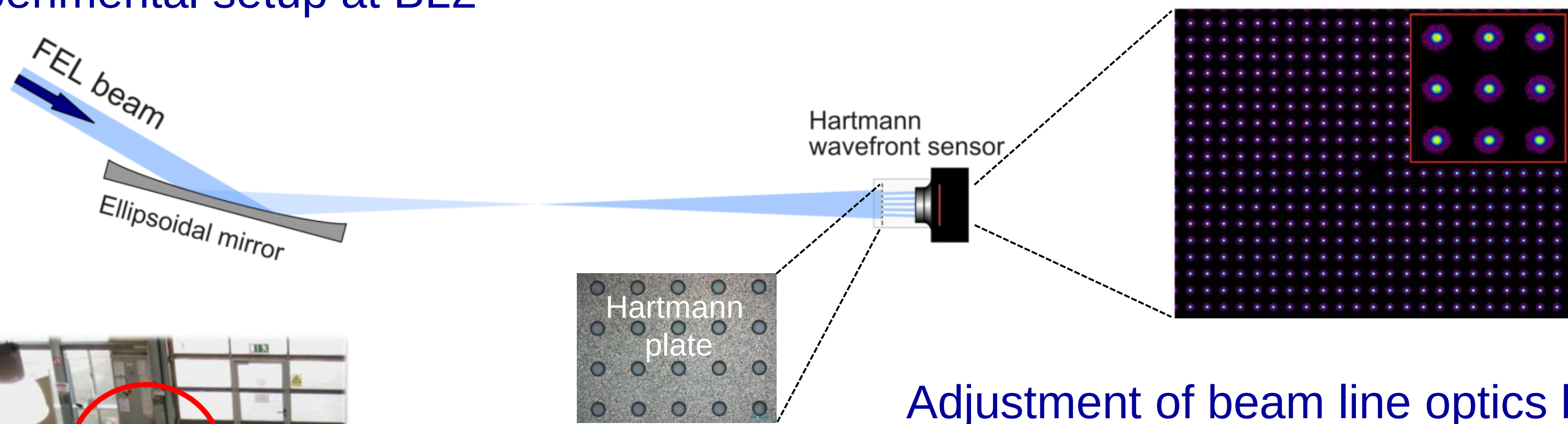
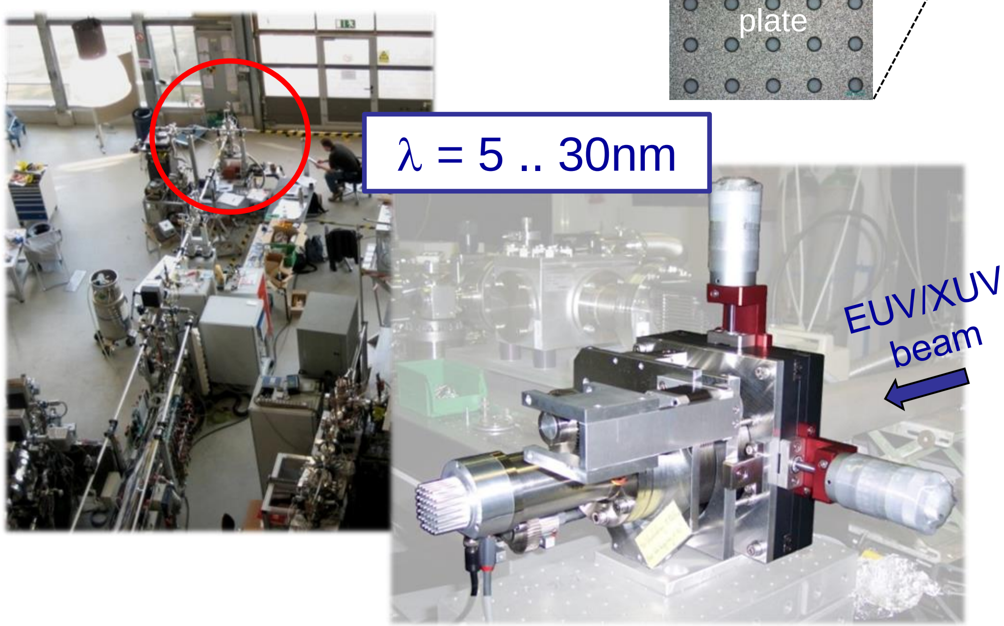
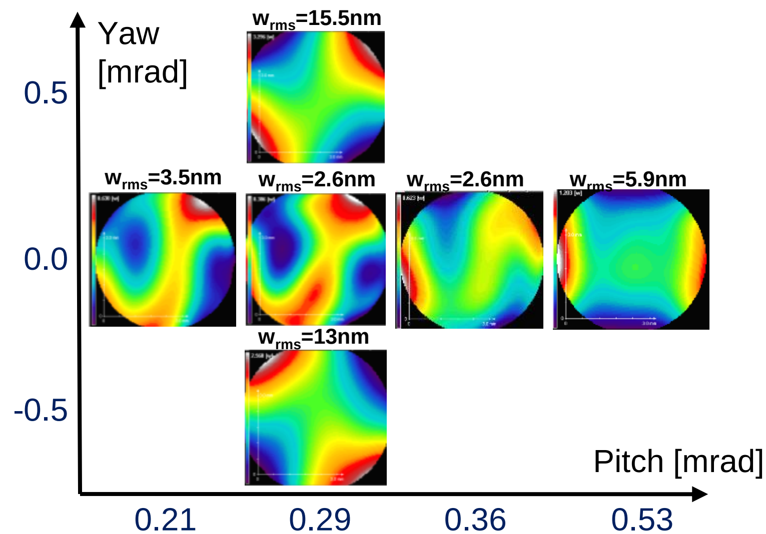
Experimental setup at BL2





Adjustment of beam line optics by online wavefront monitoring



[1] B. Flöter *et al*, *Beam parameters of FLASH beamline BL1 from Hartmann wavefront measurements*, Nucl. Instrum. Meth. A 635 (2011) 5108-5112

Beam characterization of FLASH

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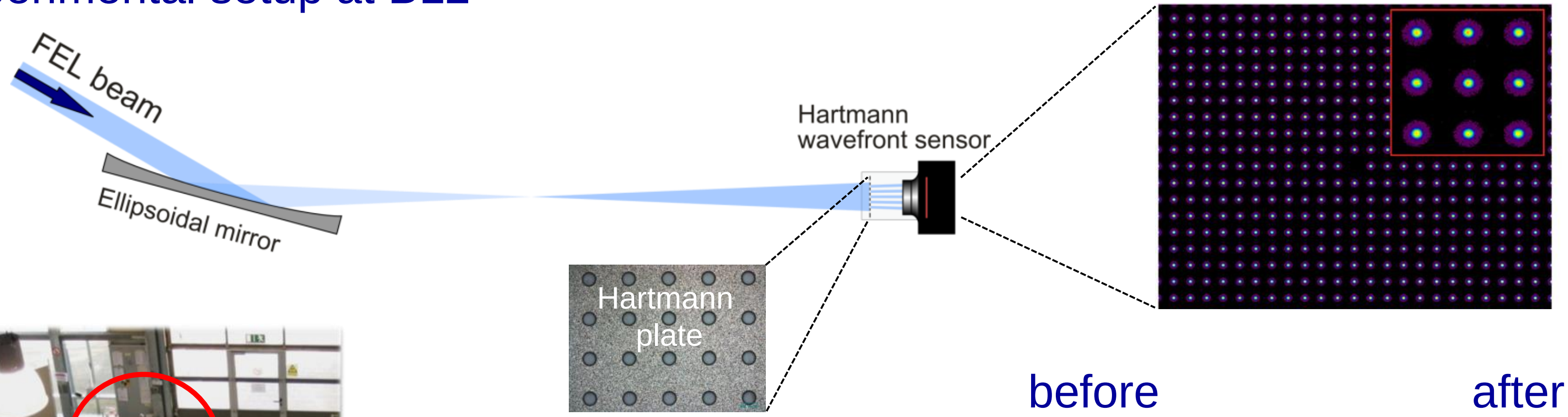
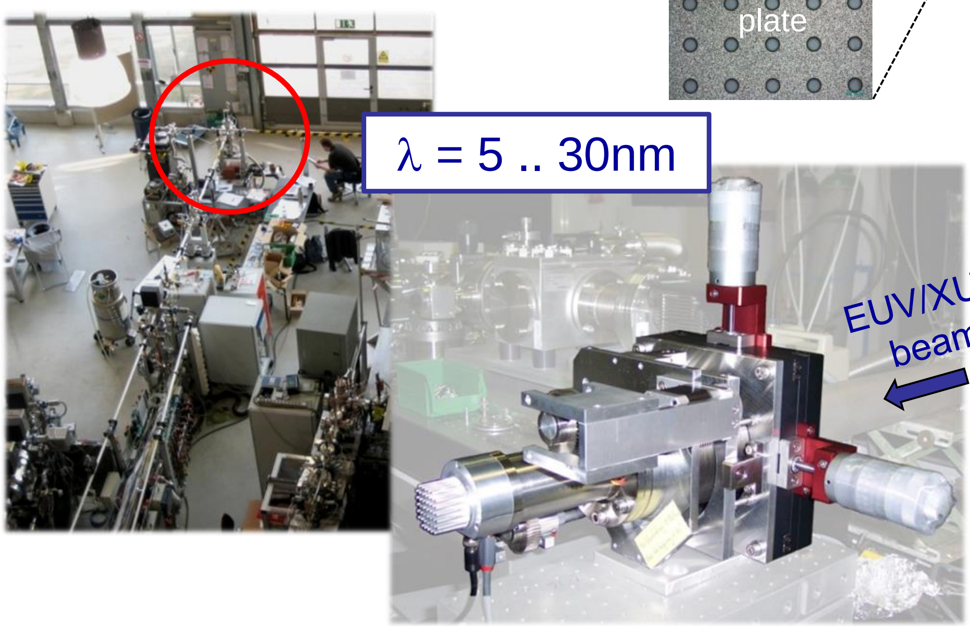
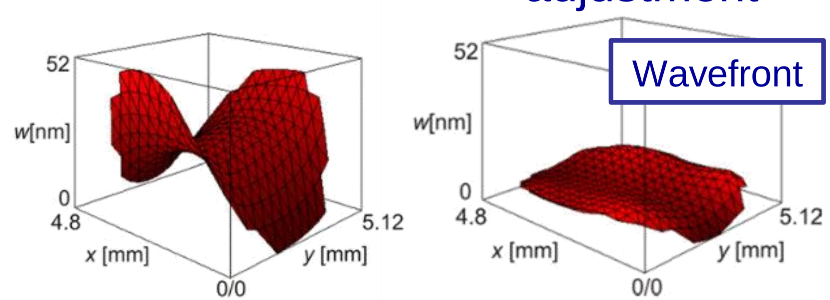
Experimental setup at BL2





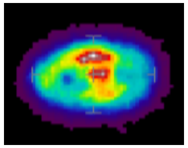
before adjustment

after adjustment

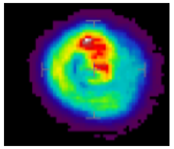


w_{rms}~ 10nm

w_{rms}~ 2.5nm ~ λ/10

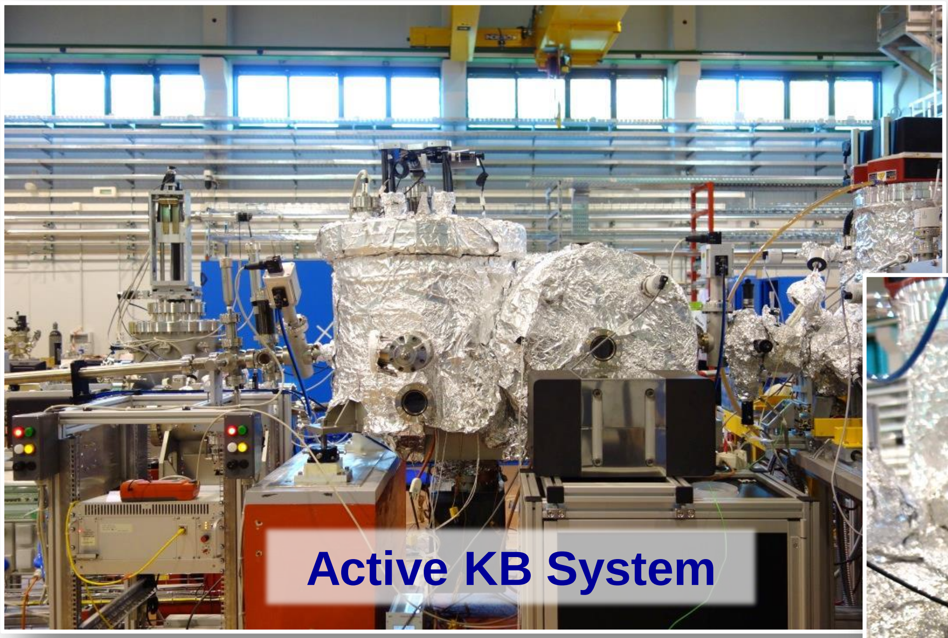


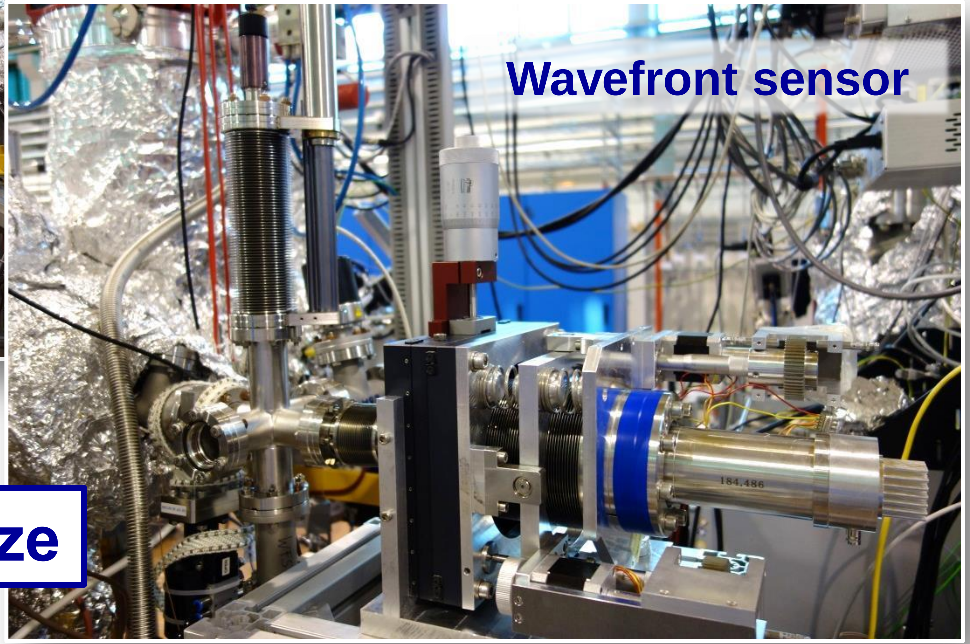
Intensity



[2] B. Flöter *et al*, *EUV Hartmann sensor for wavefront measurements at the Free-electron LASer in Hamburg*, New J. Phys. 12 (2010) 083015

Beam characterization at FERMI



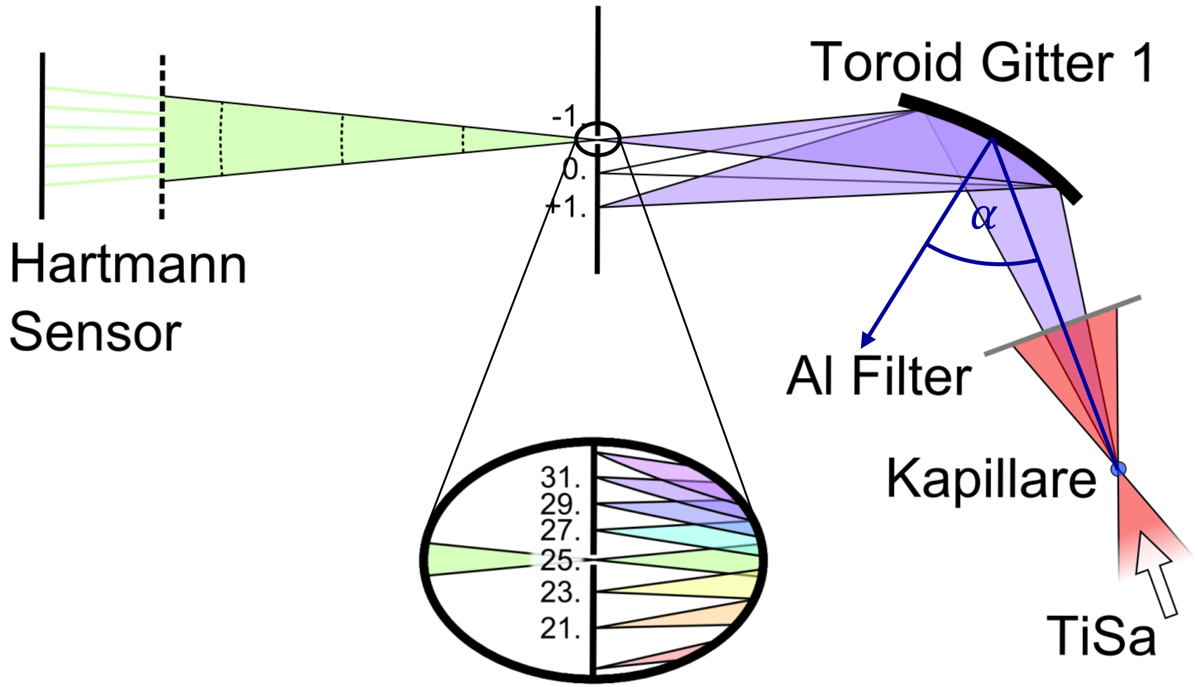


→ 10µm x 10µm focal size

Beam characterization of HHG

Adjustment of toroidal grating

- Titanium-Saphire Laser
  λ = 800 nm
  T = 40 fs
  P = 500 mW
  f = 1 kHz
- exposure time
  t = 40 s
- 25th harmonic
  λ = 32 nm
- variation of
  yaw angle α



Beam characterization of HHG

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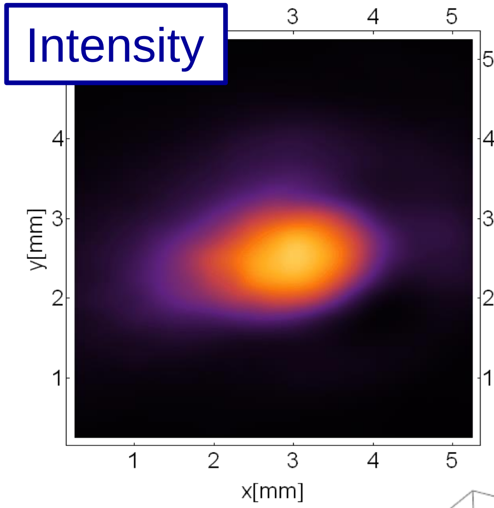
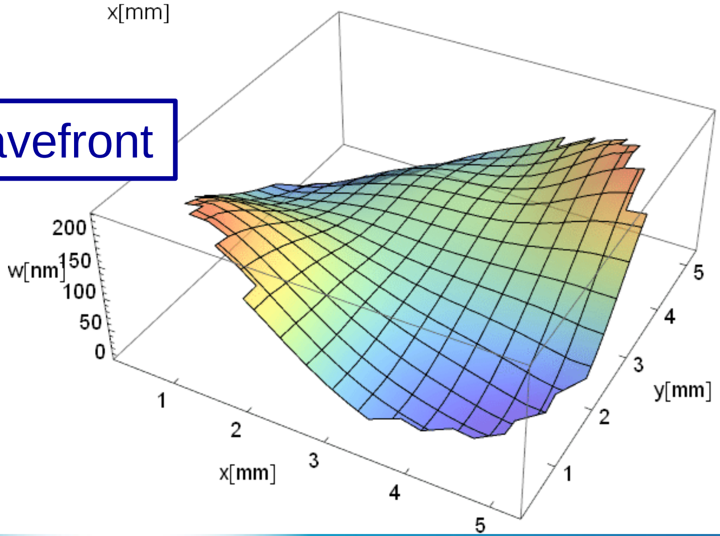
Intensity

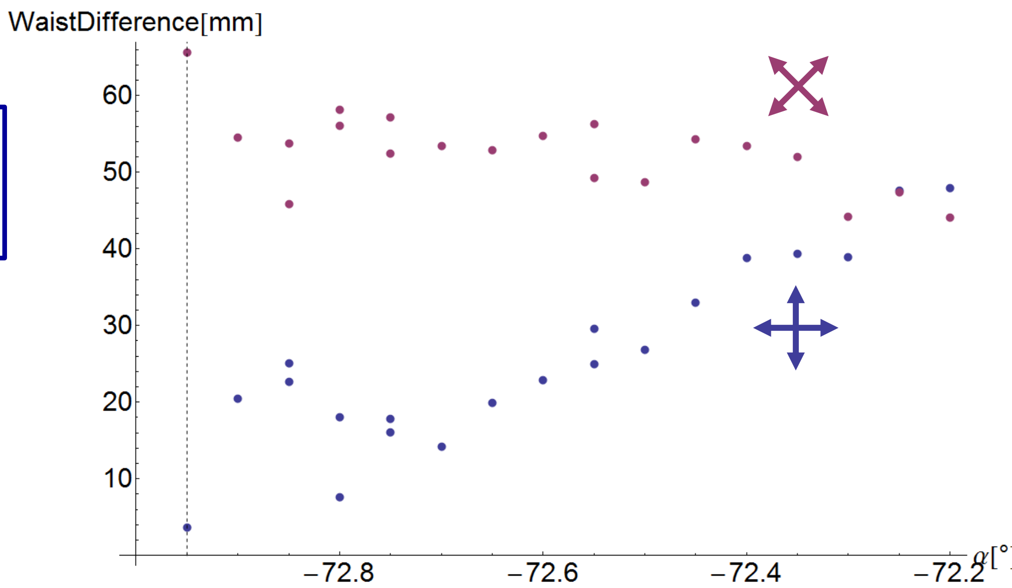


$\alpha = -72.95°$

Wavefront



Astigmatic waist difference



Beam characterization of HHG

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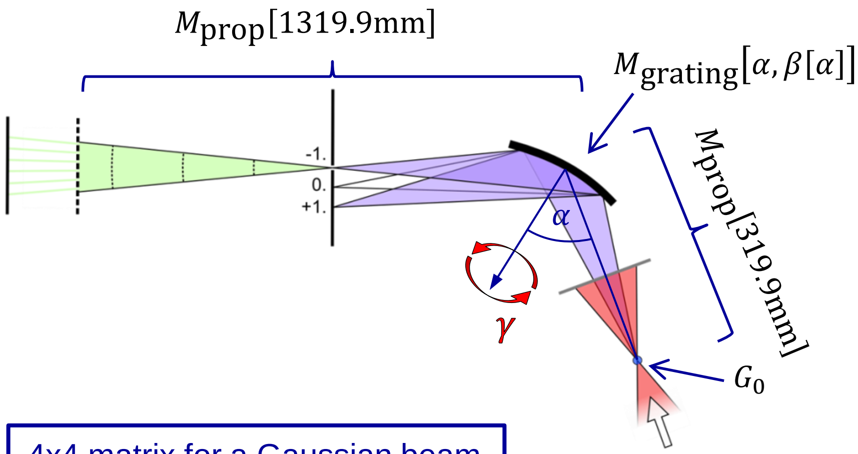


Beam propagation

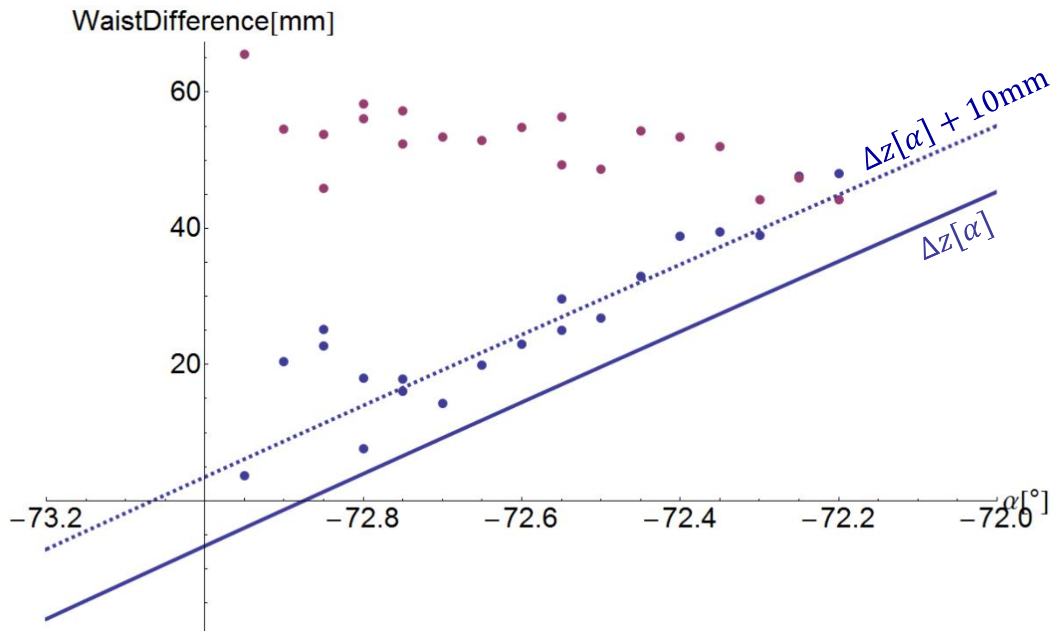
$$G[\alpha] = M_{prop}[1319.9mm] \cdot \underset{\uparrow\ M_{rot}[\gamma]}{} M_{grating}[\alpha, \beta[\alpha]] \underset{\uparrow\ M_{rot}[\gamma]^T}{} \cdot M_{prop}[319.9mm] \cdot G_0$$

4x4 matrix for a Gaussian beam

$$G = \begin{pmatrix} \langle x^2 \rangle & 0 & \langle xu \rangle & 0 \\ 0 & \langle y^2 \rangle & 0 & \langle yv \rangle \\ \langle xu \rangle & 0 & \langle u^2 \rangle & 0 \\ 0 & \langle yv \rangle & 0 & \langle v^2 \rangle \end{pmatrix}$$

4x4 matrix for a toroidal grating [3]

$$M_{grating}[\alpha, \beta] = \begin{pmatrix} M & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2/R_t & 0 & 1/M & 0 \\ 0 & -2/R_s & 0 & 1 \end{pmatrix}$$



[3] A.E. Siegman, *ABCD-matrix elements for a curved diffraction grating,* J. Opt. Am. A 2 (1985) 1793

Coherence

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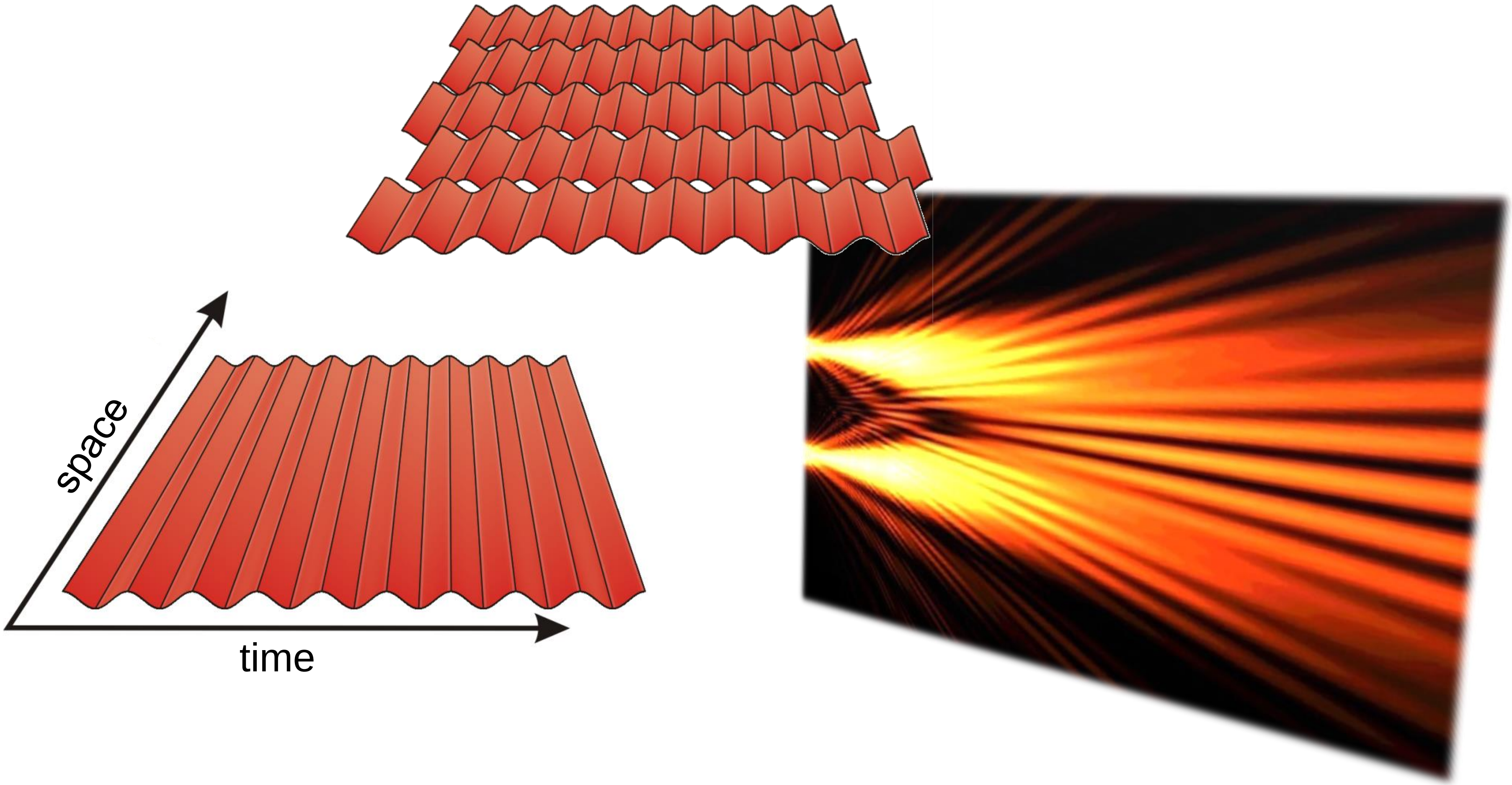
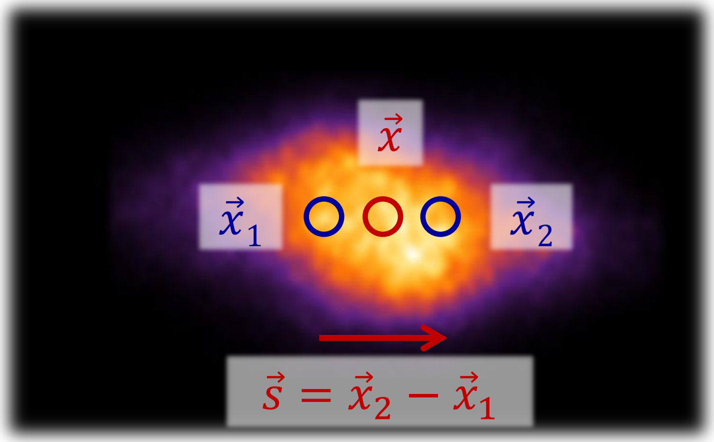


Mutual coherence function



Mutual coherence function

$$\Gamma(\vec{x},\vec{s}) = \langle E(\vec{x}_1,t)\cdot E^*(\vec{x}_2,t)\rangle$$
$$= \langle E(\vec{x}-\vec{s}/2,t)\cdot E^*(\vec{x}+\vec{s}/2,t)\rangle$$

Local degree of coherence

$$\gamma(\vec{x},\vec{s}) = \frac{\Gamma(\vec{x},\vec{s})}{\sqrt{I(\vec{x}-\vec{s}/2)\cdot I(\vec{x}+\vec{s}/2)}}$$

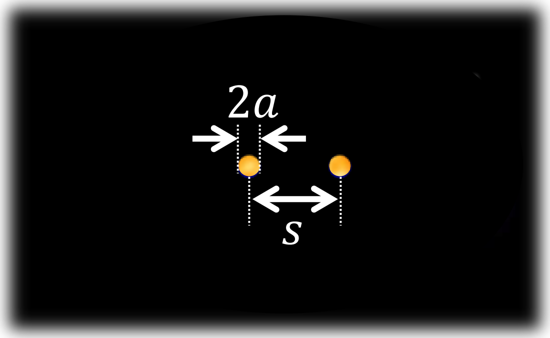
Global degree of coherence

$$K = \frac{\iint \Gamma(\vec{x},\vec{s})^2 d\vec{x}d\vec{s}}{\left(\iint \Gamma(\vec{x},0)d\vec{x}\right)^2}$$

→ ability for constructive / destructive interference

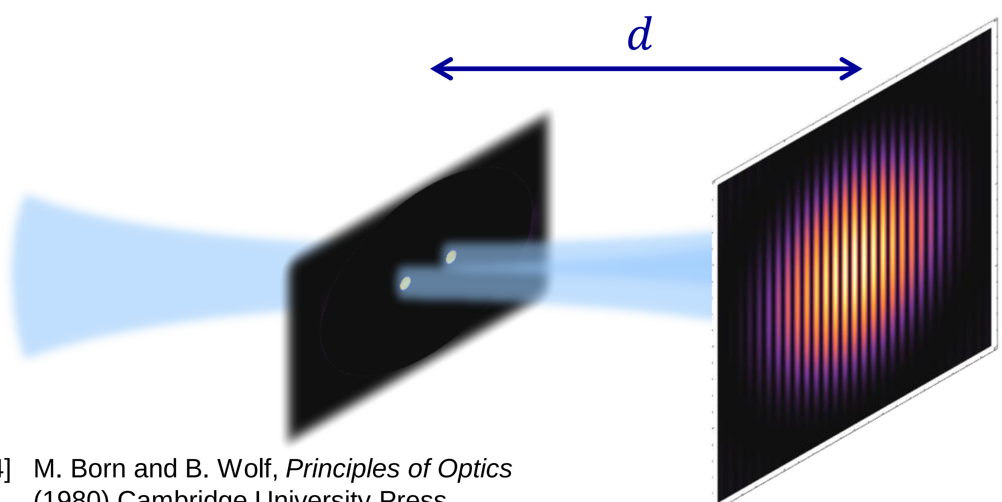
Mutual coherence function

Interference of elementary waves → $\gamma(\vec{x},\vec{s})$

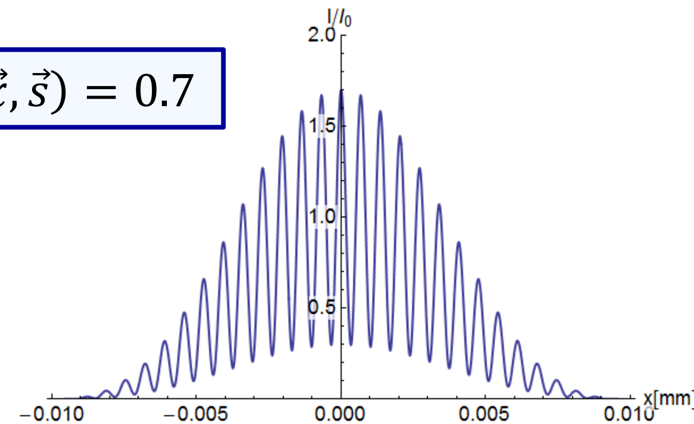


$$I(x,y) = I_0 \cdot \left(J_1\left(\frac{2\pi a r}{\lambda d}\right) \Big/ \frac{2\pi a r}{\lambda d} \right)^2 \cdot \left[1 + \gamma(\vec{x},\vec{s}) \cdot \cos\left(\frac{2\pi s}{\lambda d} x\right)\right]$$ [1]

$$r = \sqrt{x^2 + y^2}$$



$$\gamma(\vec{x},\vec{s}) = 0.7$$



[4] M. Born and B. Wolf, *Principles of Optics* (1980) Cambridge University Press

Mutual coherence function

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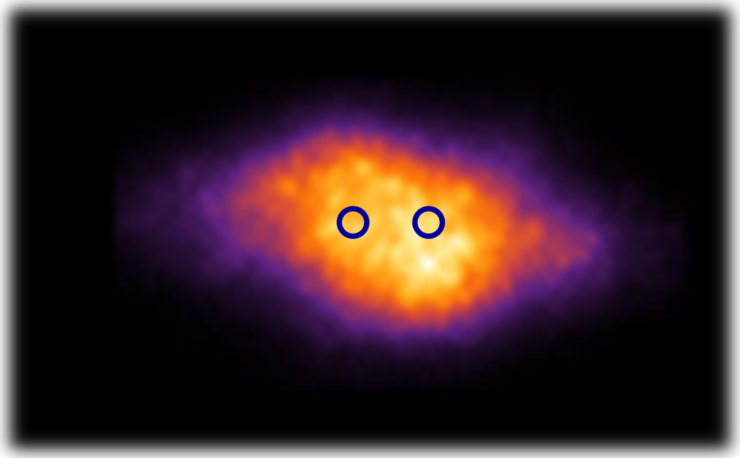
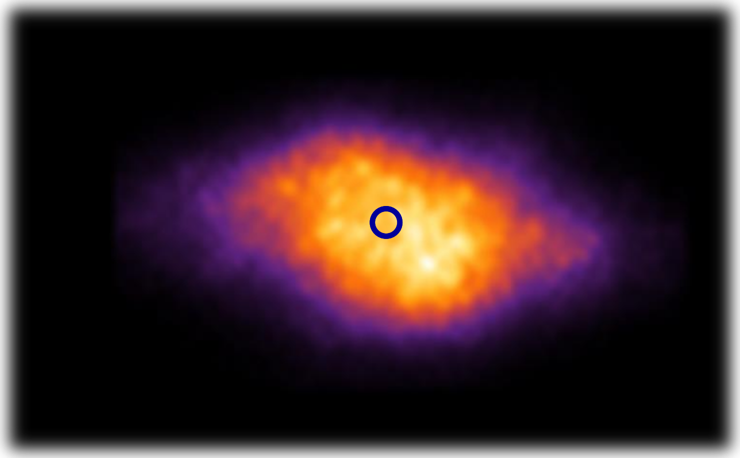
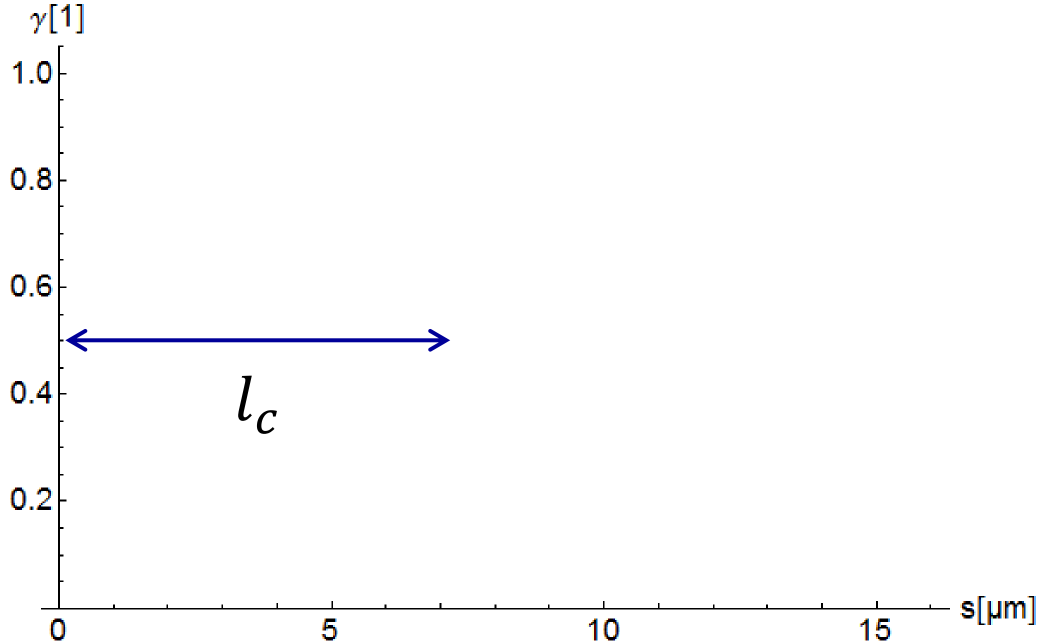


$$\gamma(0, s_x) = \frac{\Gamma(0, s_x)}{\sqrt{I(-s_x/2) \cdot I(s_x/2)}}$$

Mutual coherence function



$$\gamma(0, s_x) = \frac{\Gamma(0, s_x)}{\sqrt{I(-s_x/2) \cdot I(s_x/2)}}$$



coherence length l_c

Mutual coherence function

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Spatial and temporal coherence properties of single free-electron laser pulses

A. Singer[1], F. Sorgenfrei[2], A. P. Mancuso[3], N. Gerasimova[1], O. M. Yefanov[1], J. Gulden[1], T. Gorniak[4,5], T. Senkbeil[4,5], A. Sakdinawat[6], Y. Liu[7], D. Attwood[7], S. Dziarzhytski[1], D. D. Mai[8], R. Treusch[1], E. Weckert[1], T. Salditt[8], A. Rosenhahn[4,5], W. Wurth[2*] and I. A. Vartanyants[1,9*]

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[8] Institut für Röntgenphysik, Georg-August-Universität Göttingen, Friedrich-Hund-Platz 1, D-37077 Göttingen, Germany
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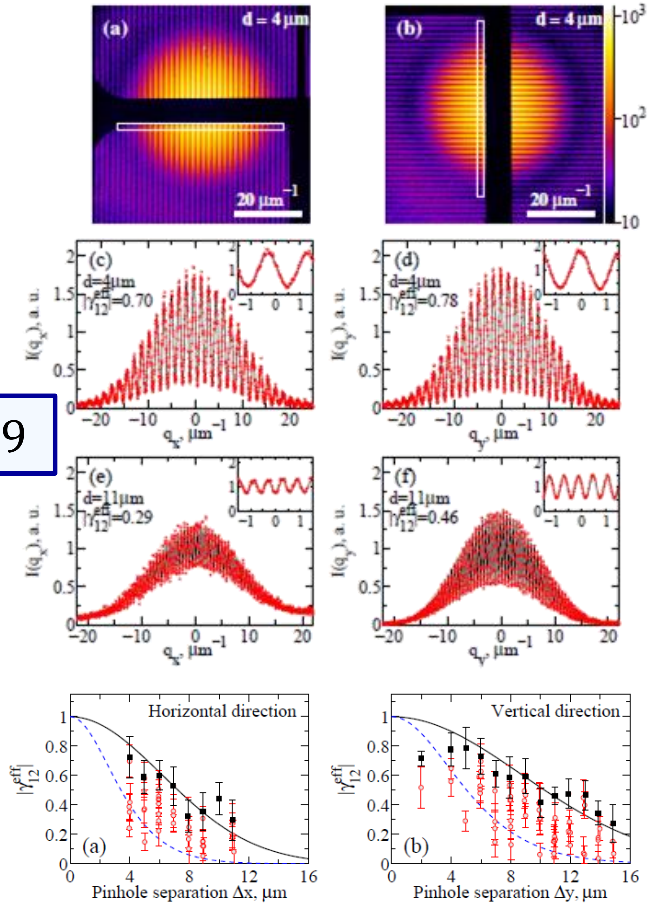
Abstract: The experimental characterization of the spatial and temporal coherence properties of the free-electron laser in Hamburg (FLASH) at a wavelength of 8.0 nm is presented. Double pinhole diffraction patterns of single femtosecond pulses focused to a size of about $10 \times 10\ \mu m^2$ were measured. A transverse coherence length of $6.2 \pm 0.9\ \mu m$ in the horizontal and $8.7 \pm 1.0\ \mu m$ in the vertical direction was determined from the most coherent pulses. Using a split and delay unit the coherence time of the pulses produced in the same operation conditions of FLASH was measured to be 1.75 ± 0.01 fs. From our experiment we estimated the degeneracy parameter of the FLASH beam to be on the order of 10^{10} to 10^{11}, which exceeds the values of this parameter at any other source in the same energy range by many orders of magnitude.

OCIS codes: (000.0000) General.

$$K = 0.42 \pm 0.09$$

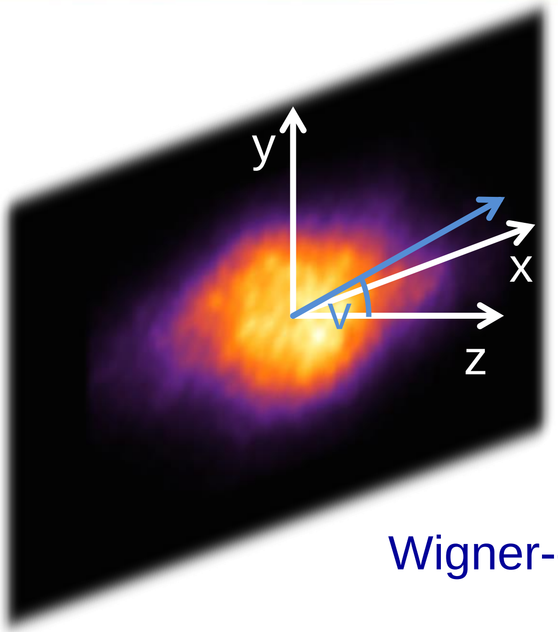


$l_h = 6.2\mu m$ $l_v = 8.7\mu m$

References and links

1. W. Ackermann, G. Asova, V. Ayvazyan, A. Azima, N. Baboi, J. Bähr, V. Balandin, B. Beutner, A. Brandt, A. Bolzmann, R. Brinkmann, O. I. Brovko, M. Castellano, P. Castro, L. Catani, E. Chiadroni, S. Choroba, A. Cianchi, J. T. Costello, D. Cubaynes, J. Dardis, W. Decking, H. Delsim-Hashemi, A. Delserieys, G. Di Pirro, M. Dohlus, S. Dsterer, A. Eckhardt, H. T. Edwards, B. Faatz, J. Feldhaus, K. Flttmann, J. Frisch, L. Fröhlich, T. Garvey, U. Gensch, C. Gerth, M. Görler, N. Golubeva, H. J. Grabosch, M. Grecki, O. Grimm, K. Hacker, U. Hahn, J. H. Han, K. Honkavaara, T. Hott, M. Htning, Y. Ivanisenko, E. Jaeschke, W. Jalmuzna, T. Jezynski, R. Kammering, V. Katalev, K. Kavanagh, E. T. Kennedy, S. Khodyachykh, K. Klose, V. Kocharyan, M. Körfer, M. Kollewe, W.

Wigner distribution function



spatial coordinate $\vec{x} = \binom{x}{y}$

mutual coherence function

$$h(\vec{x},\vec{u}) = \left(\frac{1}{2\pi}\right)^2 \cdot \iint \Gamma(\vec{x},\vec{s}) \cdot e^{-i\vec{u}\cdot\vec{s}} d^2s$$

Wigner-distribution

angular coordinate $\vec{u} = \binom{u}{v}$

Interpretation: radiance at position $\vec{x}$ in direction of $\vec{u}$

$[h] = {}^{\mathrm{W}}/_{\mathrm{m}^2 \cdot \mathrm{sr}}$

[6] M. J. Bastiaans, *Wigner distribution function and its application to first-order optics*
J. Opt. Soc. Am. 69 (1979) 1710-1716

Wigner distribution function

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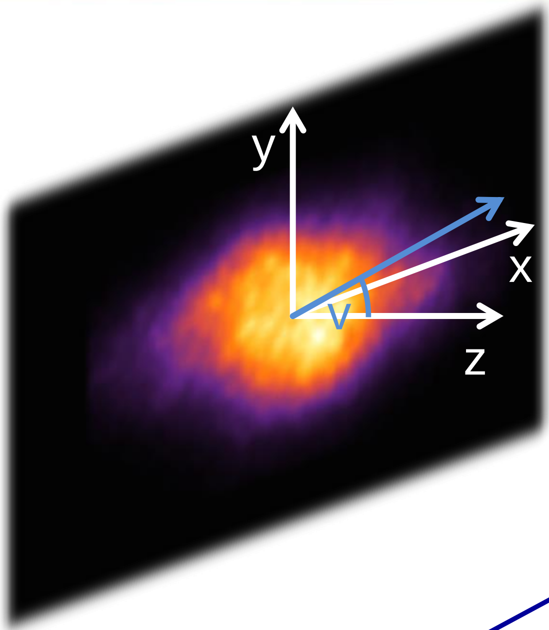


$$h(\vec{x},\vec{u}) = \left(\frac{1}{2\pi}\right)^2 \cdot \iint \Gamma(\vec{x},\vec{s}) \cdot e^{-i\vec{u}\cdot\vec{s}} d^2 s$$

Irradiance profile

$$I(\vec{x}) = \iint h(\vec{x},\vec{u}) du dv$$

$\rightarrow$ near field

Radiant intensity

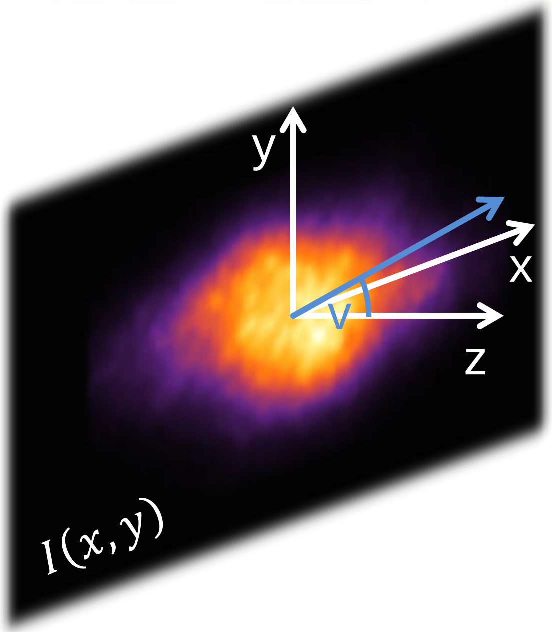
$$\tilde{I}(\vec{u}) = (2\pi)^{-2} \iint h(\vec{x},\vec{u}) dx dy$$

$\rightarrow$ far field

Degree of coherence

$$K = \frac{\iint h(\vec{x},\vec{u})^2 dx^2 du^2}{\iint h(\vec{x},\vec{u}) dx^2 du^2}$$

Wigner distribution function



Wigner distribution

$$h(\vec{x},\vec{u}) = \left(\frac{1}{2\pi}\right)^2 \cdot \iint \Gamma(\vec{x},\vec{s}) \cdot e^{-i\vec{u}\cdot\vec{s}} d^2 s$$

Fourier transform

$$\tilde{h}(\vec{w},\vec{t}) = \iint h(\vec{x},\vec{u}) \cdot e^{i\vec{x}\cdot\vec{w}} e^{i\vec{u}\cdot\vec{t}}\, d^2x\, d^2u$$

Relation to intensity distribution

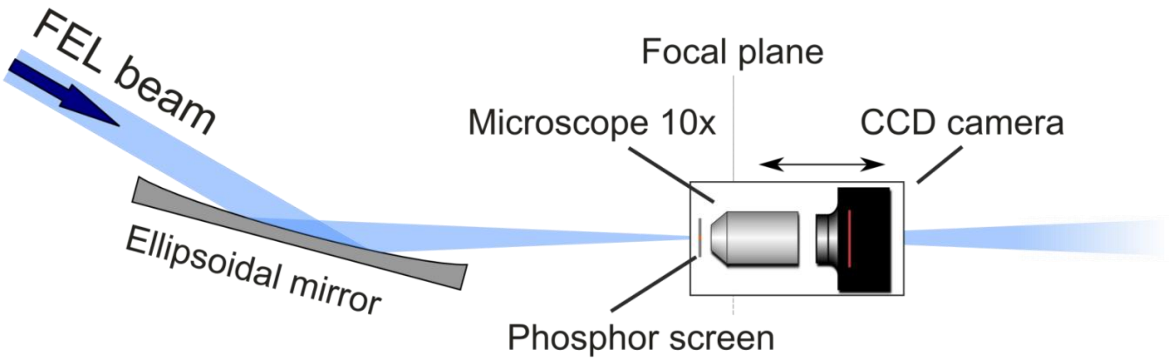
$$\tilde{h}(\vec{w}, z\cdot\vec{w}) = \tilde{I}_z(\vec{w})$$

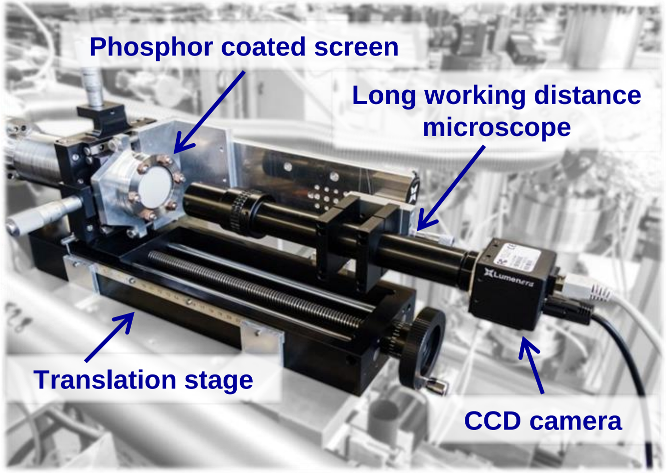
separable $\leftrightarrow$ $I(x,y) = I(x)\cdot I(y)$

$$\tilde{h}_x(w_x, z\cdot w_x) = \tilde{I}_z(w_x)$$

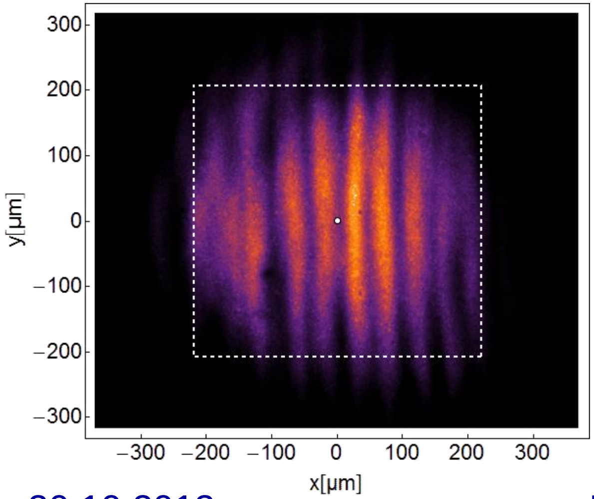
$$\tilde{h}_y(w_y, z\cdot w_y) = \tilde{I}_z(w_y)$$

Measurement at FLASH

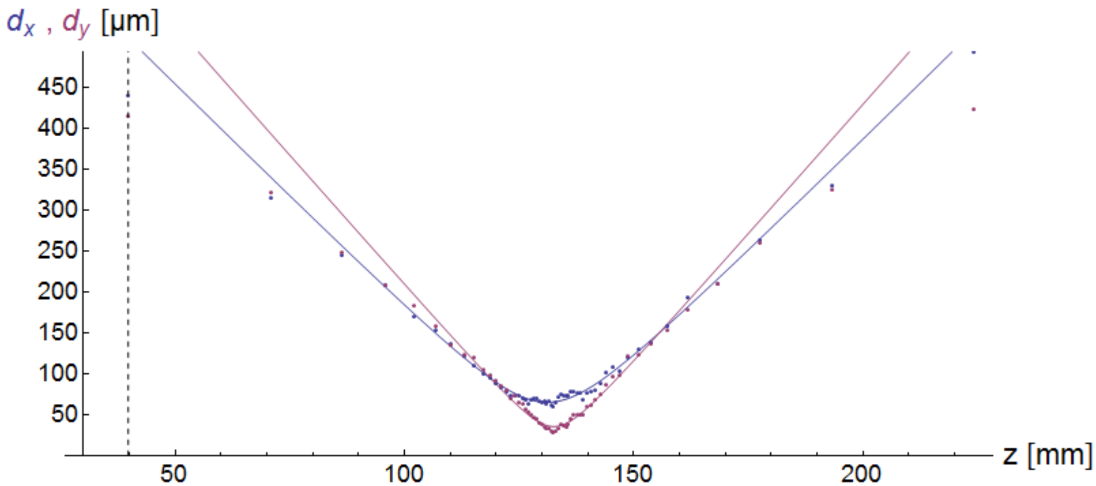




Intensity distribution



Beam diameter



Measurement at FLASH

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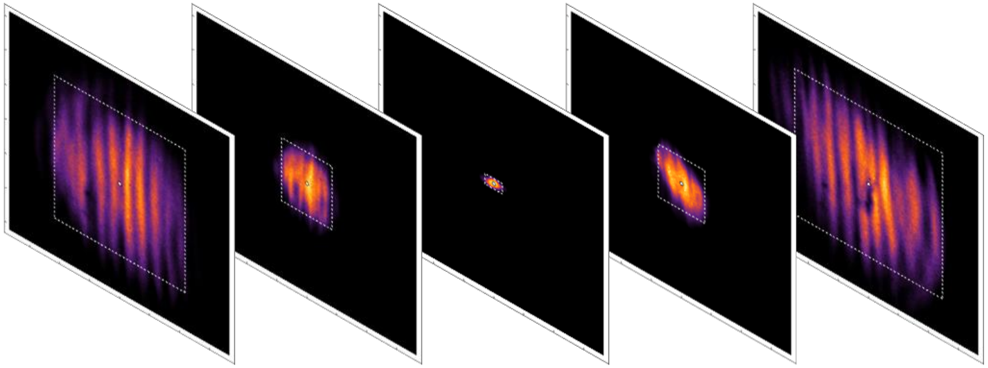

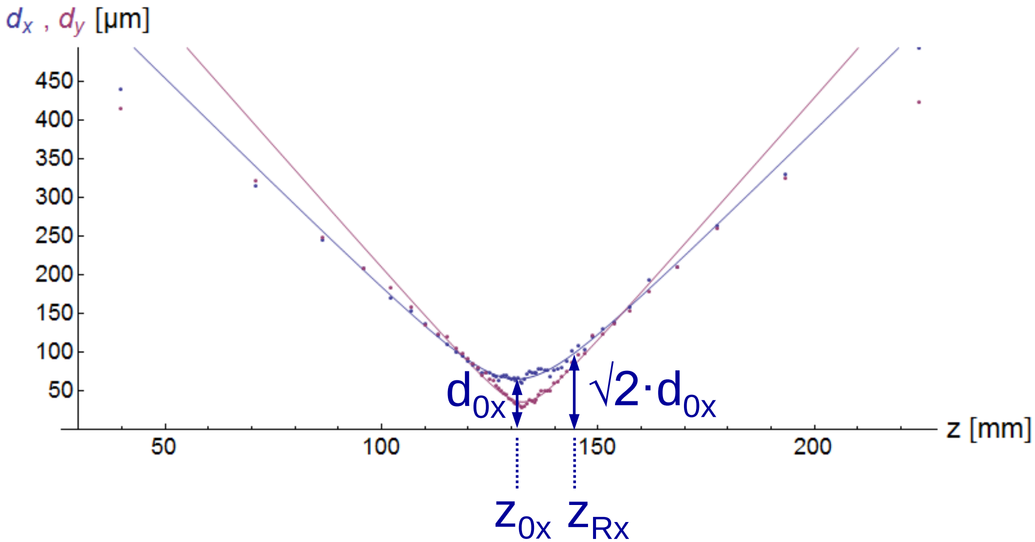

Beam parameter	Value
z_{0x} / z_{0y} [mm]	131.1 / 132.6
d_{0x} / d_{0y} [μm]	65.5 / 35.9
z_{Rx} / z_{Ry} [mm]	11.8 / 5.7
M^2_x / M^2_y [1]	21 / 13
coherence	???

Reconstruction of Wigner distribution

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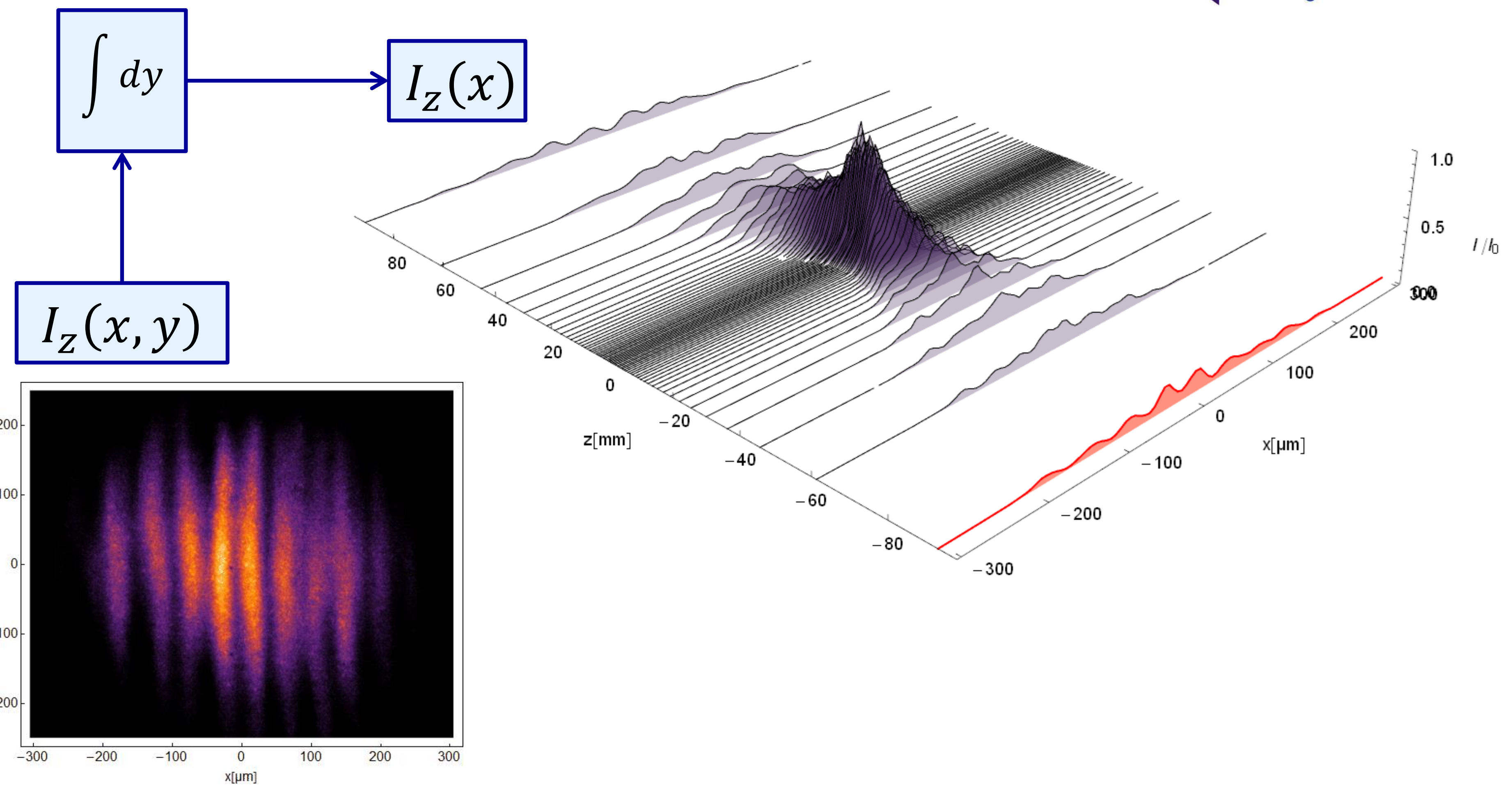
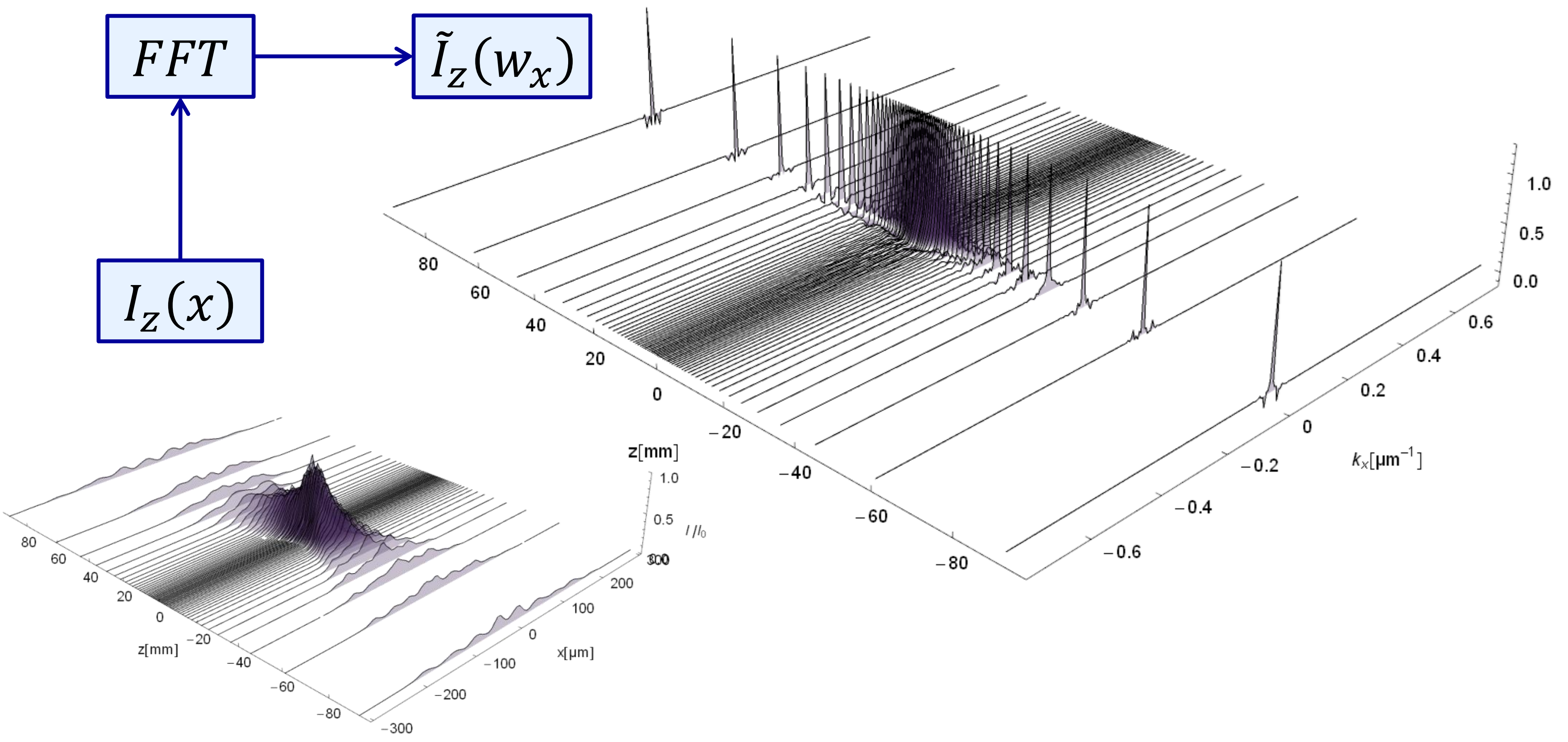


Reconstruction of Wigner distribution



Reconstruction of Wigner distribution

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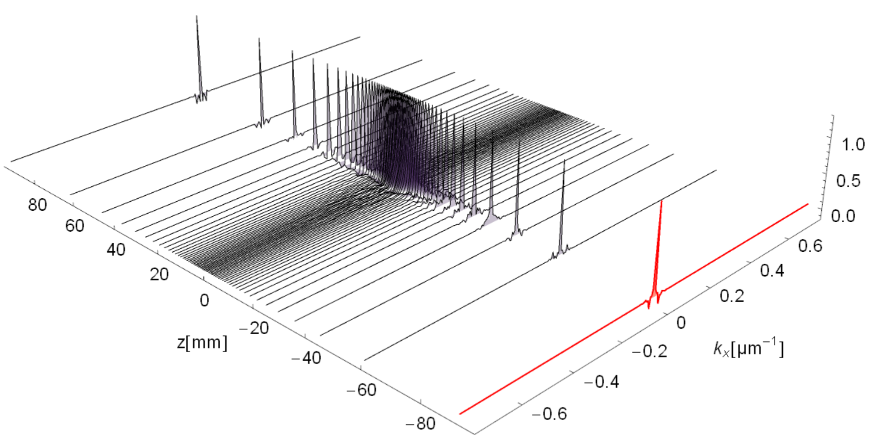
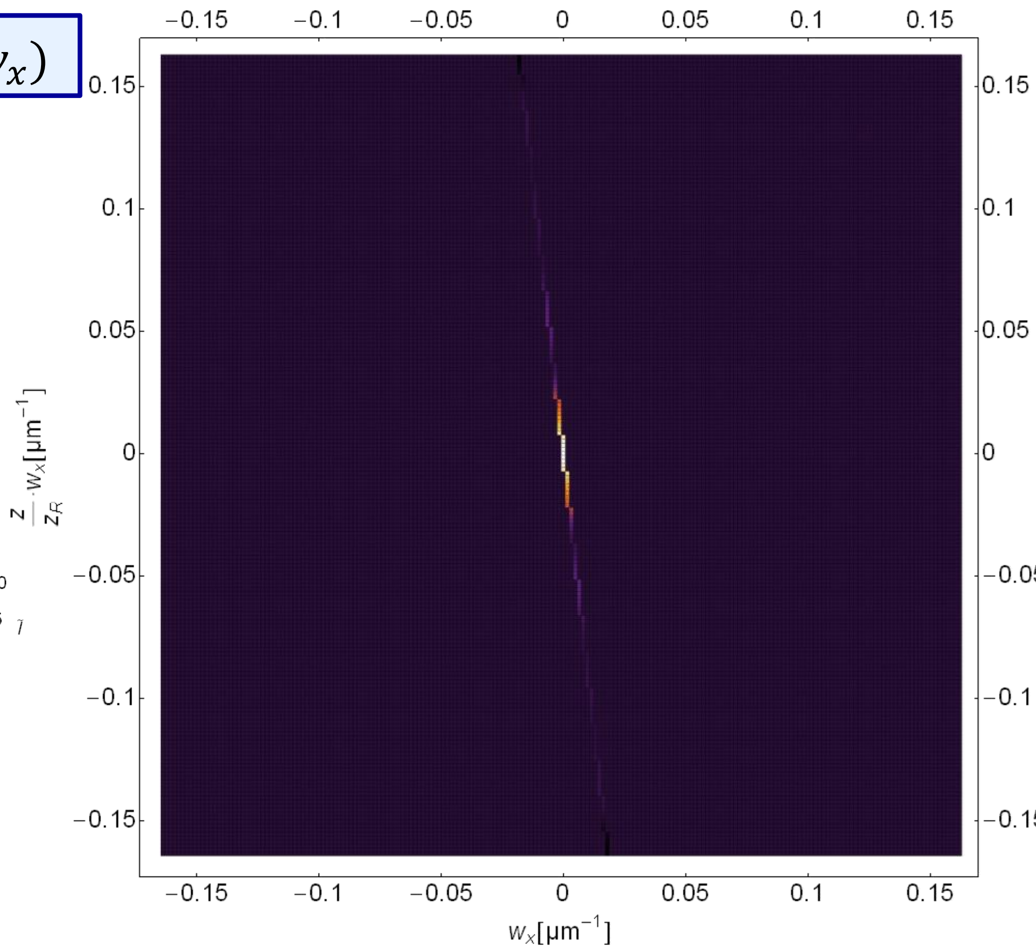
$\tilde{I}_z(w_x) = \tilde{h}(w_x, z \cdot w_x)$ → $\tilde{h}(w_x, z \cdot w_x)$

↑

$\tilde{I}_z(w_x)$





Wigner distribution of FLASH

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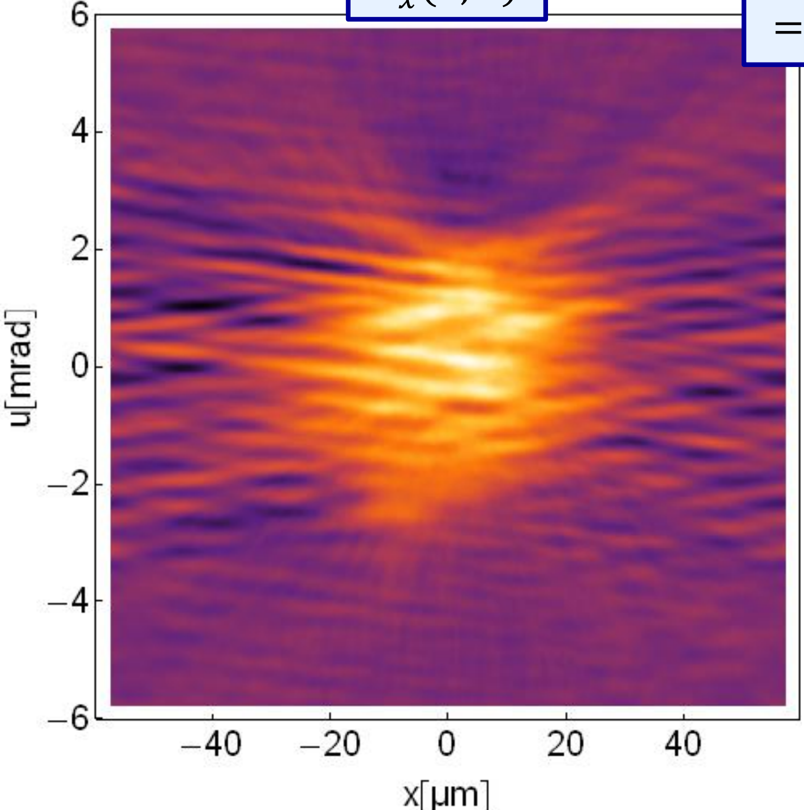
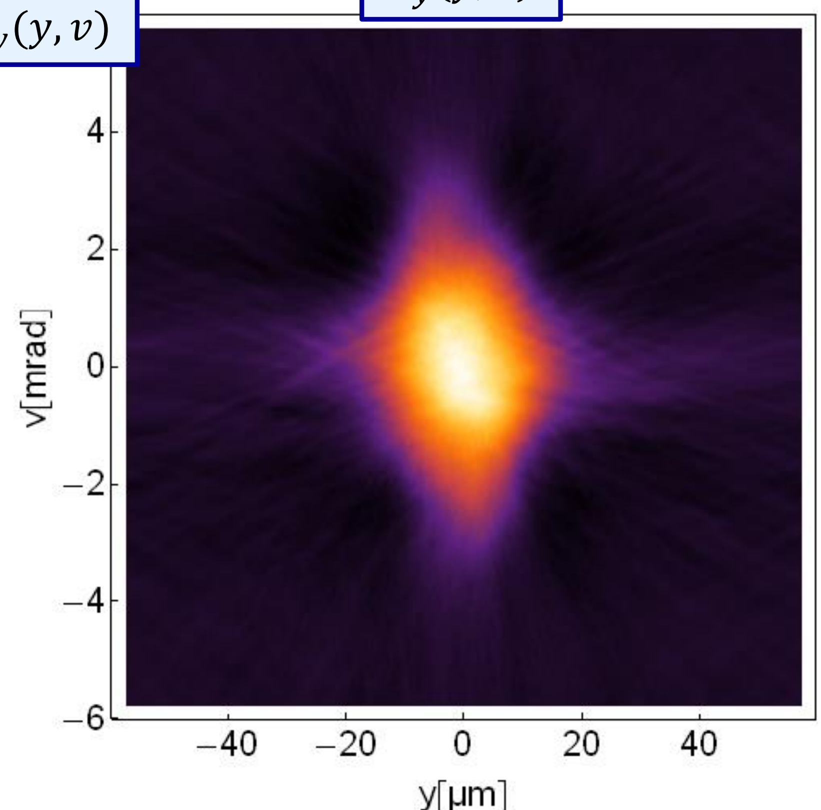
2D reconstruction (2·180² pixel = 4MB)

$$h(x,y,u,v) = h_x(x,u) \cdot h_y(y,v)$$

$h_x(x,u)$



$h_y(y,v)$



$$K_x = \frac{\iint h_x(x,u)^2 dxdu}{\iint h_x(x,u) dxdu} = 6.5\%$$

$$K_y = \frac{\iint h_y(y,v)^2 dydv}{\iint h_y(y,v) dydv} = 10.9\%$$

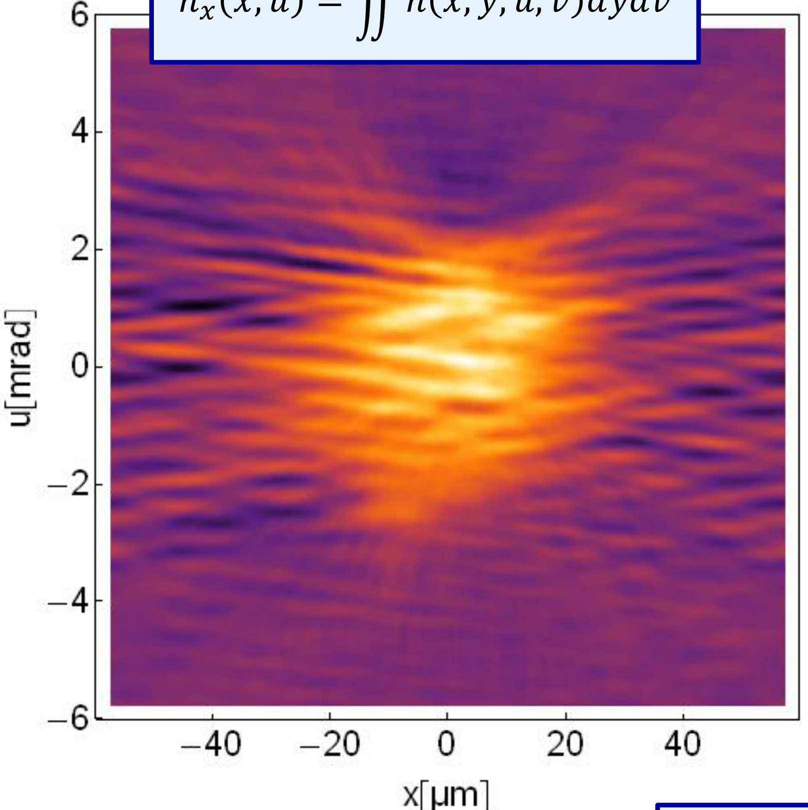
Wigner distribution of FLASH

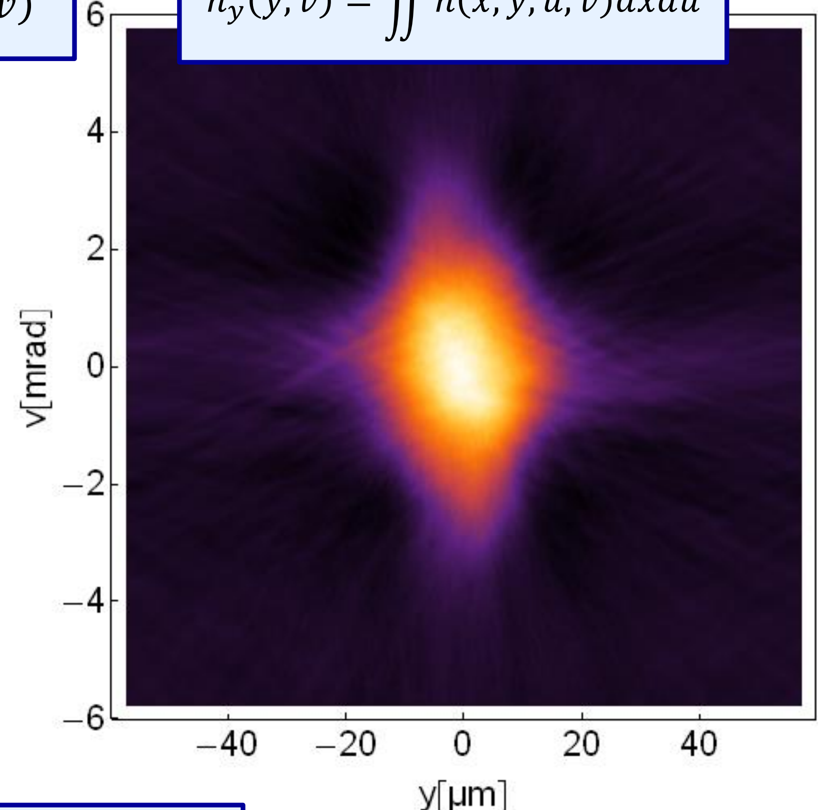
4D reconstruction (180^4 Voxel = 15 GB)

$$h_x(x,u) = \iint h(x,y,u,v)\,dy\,dv$$

$$h(x,y,u,v)$$

$$h_y(y,v) = \iint h(x,y,u,v)\,dx\,du$$

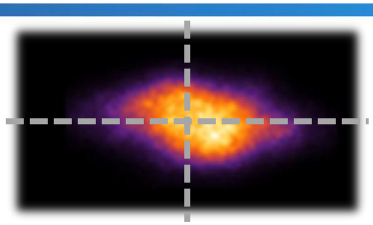




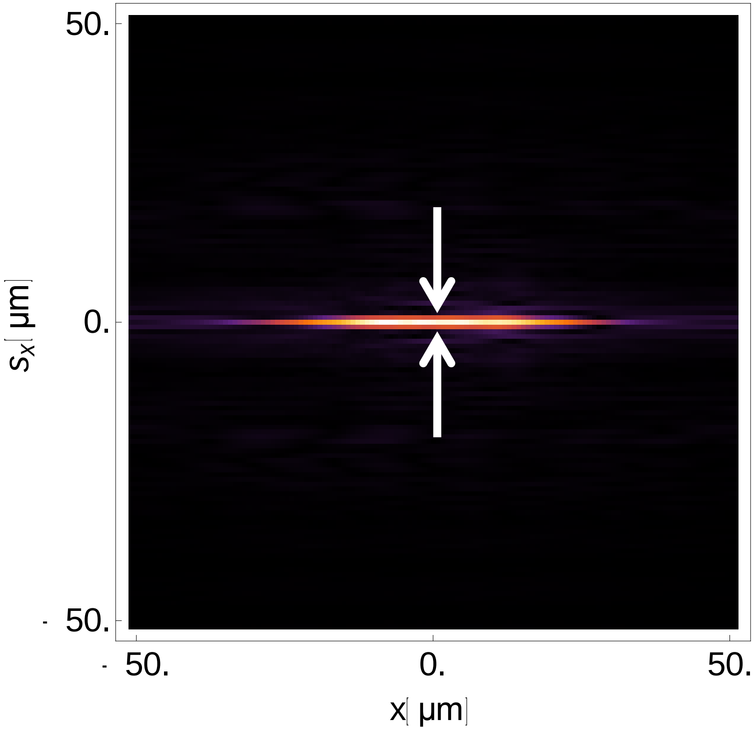
$$K = \frac{\iint h(\vec{x},\vec{u})^2 dx^2 du^2}{\iint h(\vec{x},\vec{u}) dx^2 du^2} = 1.6\%$$

Wigner distribution of FLASH

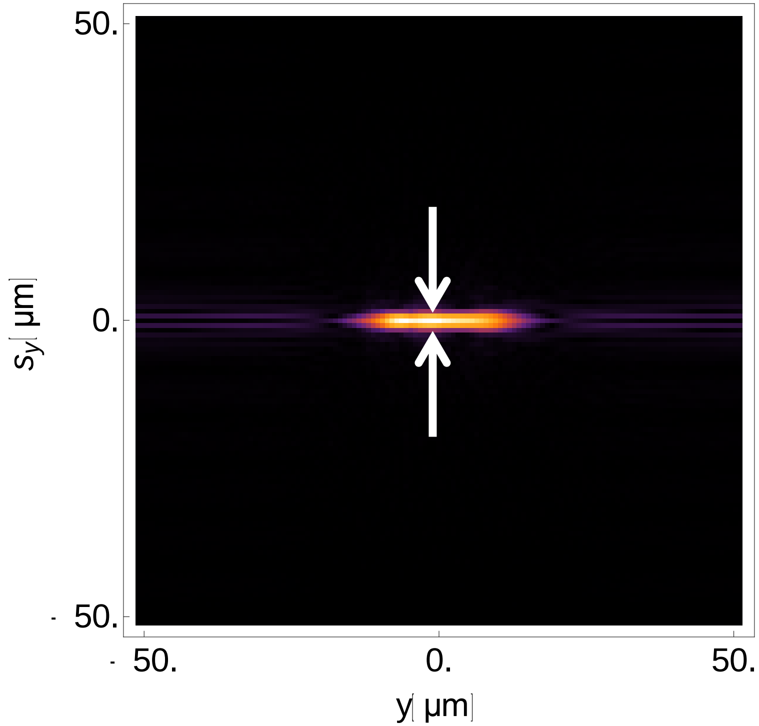
Mutual coherence function $\Gamma(\vec{x},\vec{s})$



$\Gamma(x, 0, s_x, 0)$



$\Gamma(0, y, 0, s_y)$



$$l_x = \mathrm{FWHM}\big(\Gamma(0,0,s_x,0)\big)$$
$$= 1.5\ \mu m$$

$$l_y = \mathrm{FWHM}\Big(\Gamma\big(0,0,0,s_y\big)\Big)$$
$$= 2.1\ \mu m$$

Wigner distribution of FLASH

Propagation (separable): $h(\vec{x},\vec{u}) = h_0(\vec{x} - z \cdot \vec{u}, \vec{u})$

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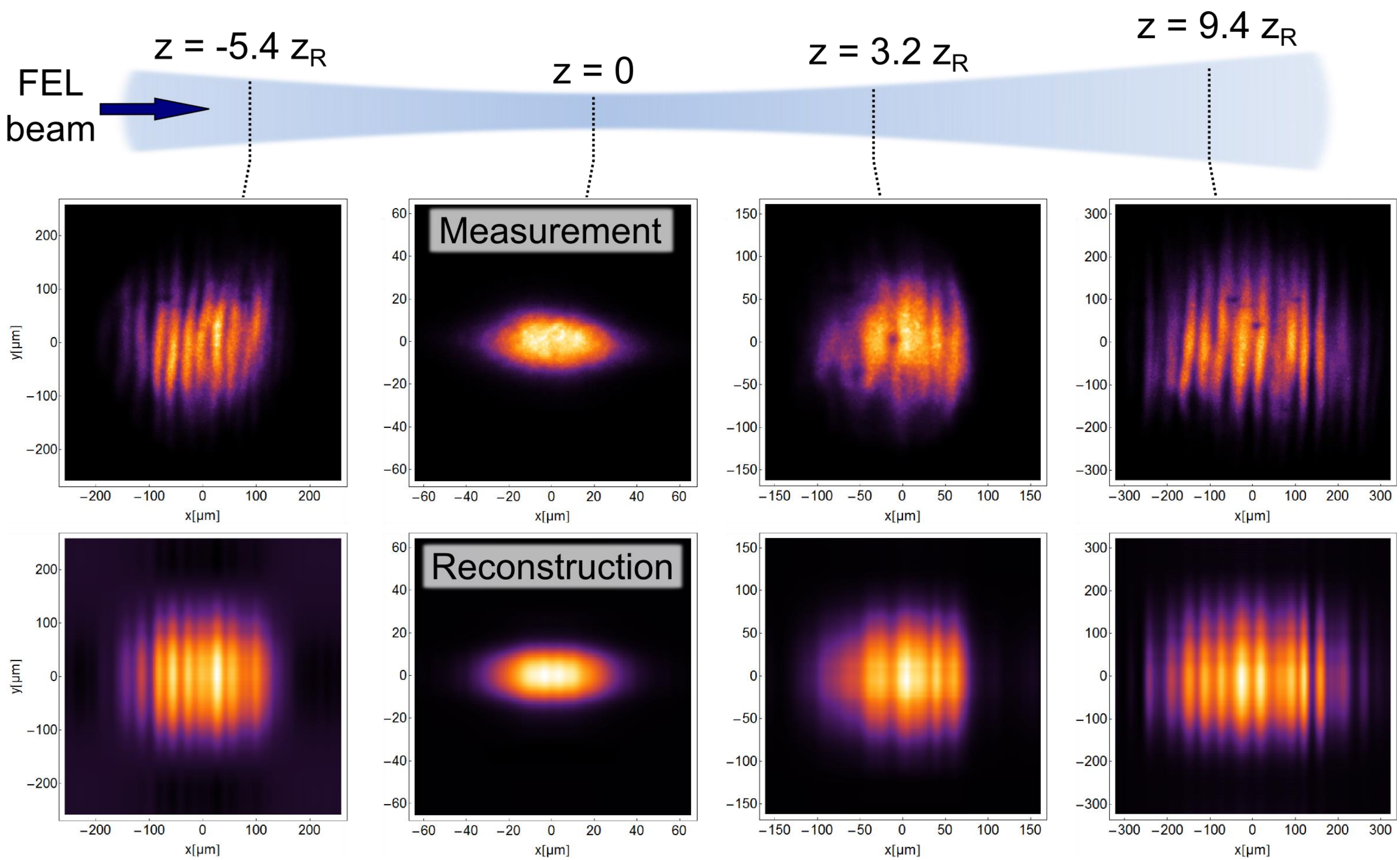
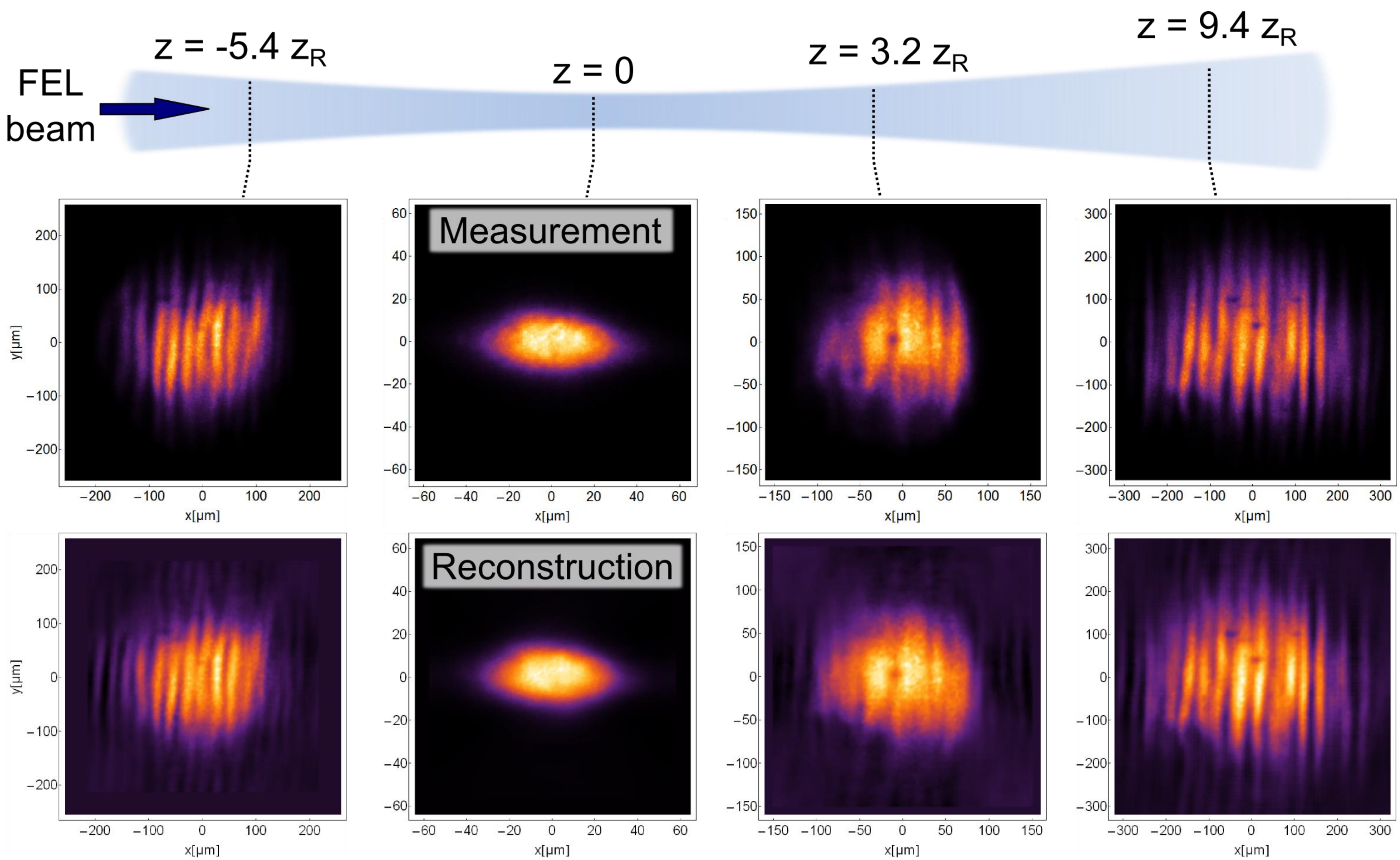


Wigner distribution of FLASH

Laser-Laboratorium Göttingen e.V.

Propagation (non-separable): $h(\vec{x},\vec{u}) = h_0(\vec{x} - z \cdot \vec{u}, \vec{u})$



Comparison of methods

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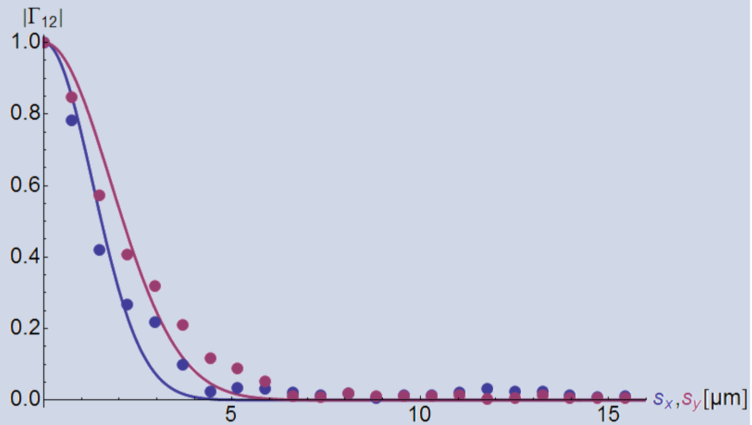
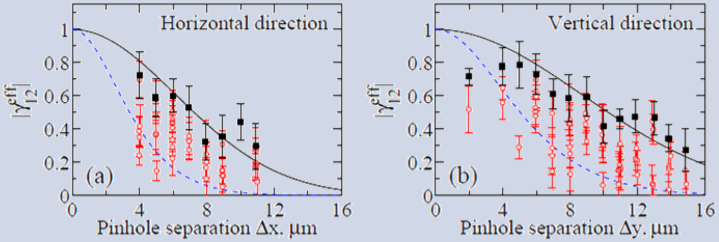
	Wigner	Young
Mutual coherence function		
Coherence lengths l_x / l_y	1.5µm / 2.1µm	6.2µm / 8.7µm
Global degree of coherence K	0.016	0.42

Comparison of methods

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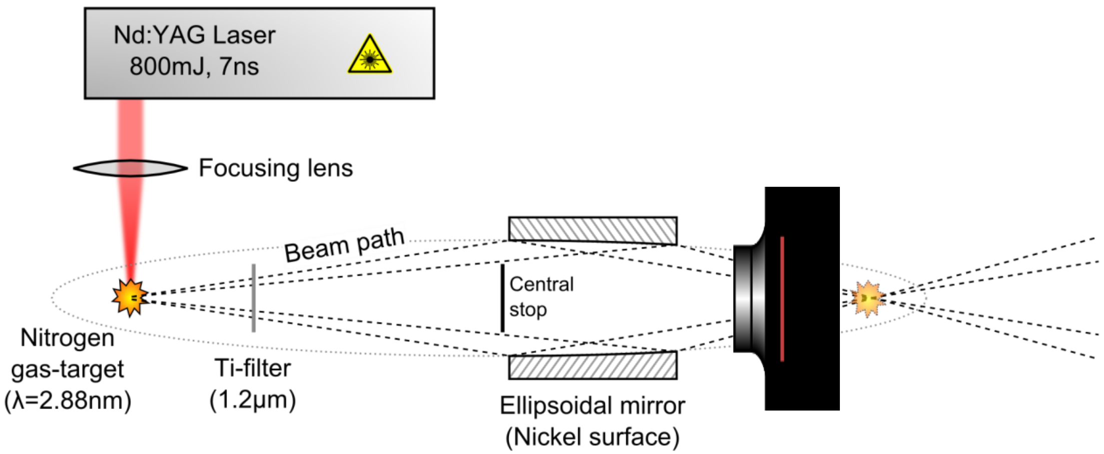
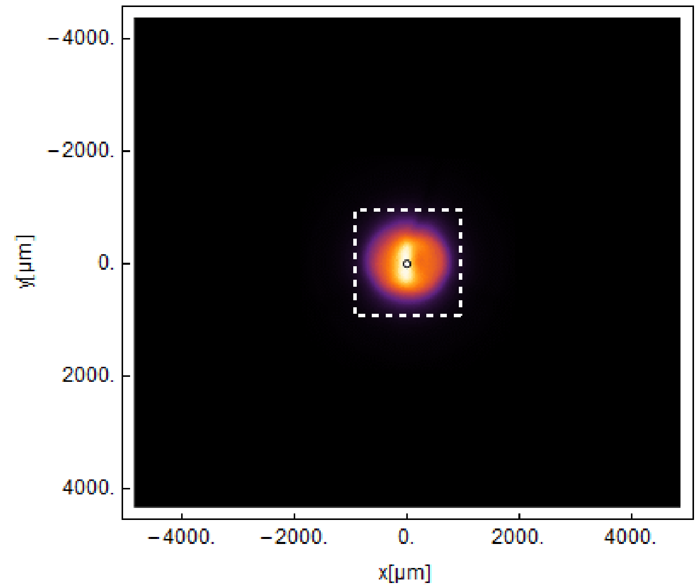
	Wigner	Young
	○ pulse to pulse fluctuations of coherence properties → *mean value* ○ huge amount of information ○ 3D measurement for a 4D phase space	○ pulse to pulse fluctuations of coherence properties & position instability → *max value* ○ comparably low information density

Wigner distribution of LPP source

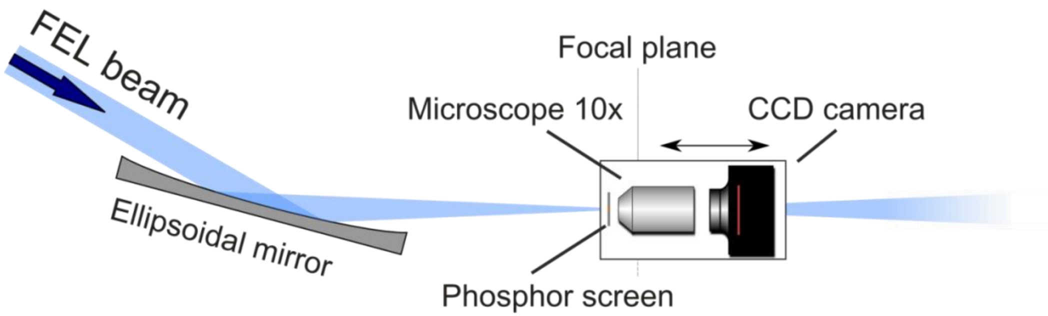
Laser produced plasma source





	x	y
Waist diameter d_0[µm]	1347	1356
Rayleigh length z_R[mm]	15.3	14.3
M^2[1]	34600	33900
Coherence K[1]	$3.2\cdot10^{-9}$	

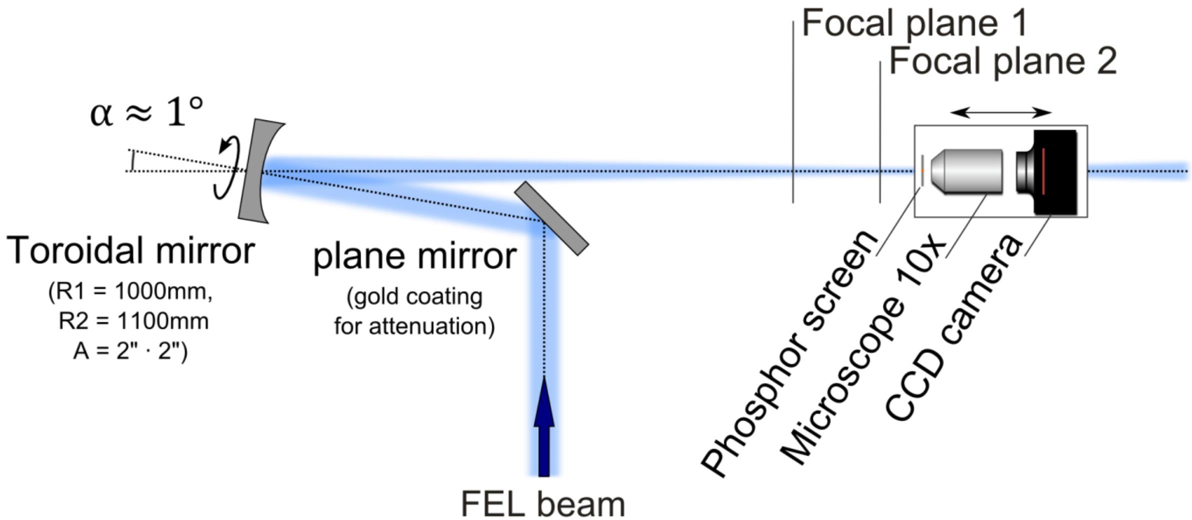
Outlook



3D measurement

4D phase space = 180^4 voxels

exp. data = 180^3 voxels

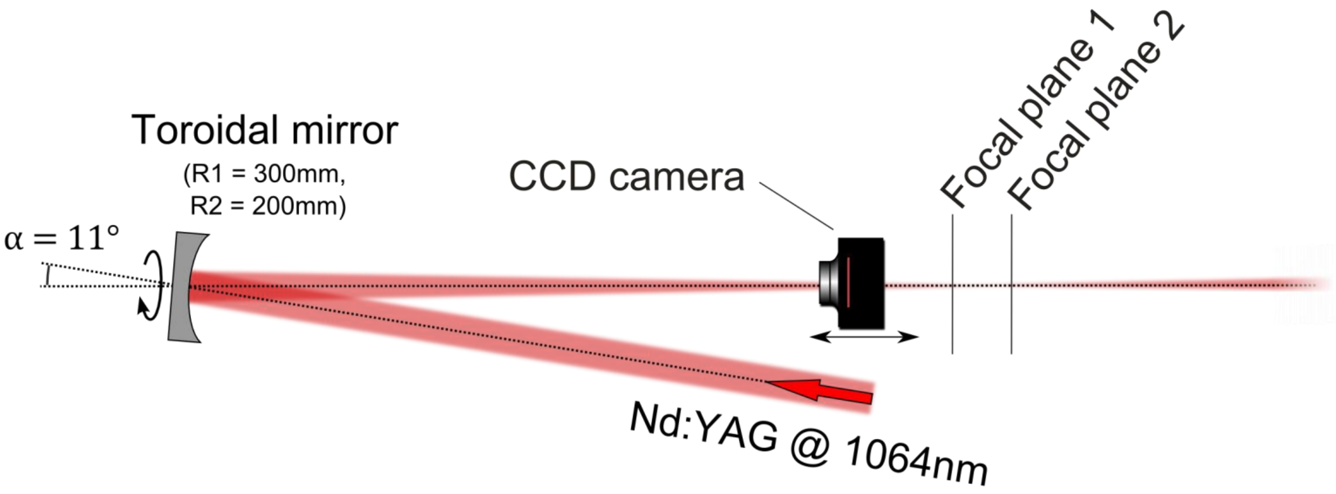


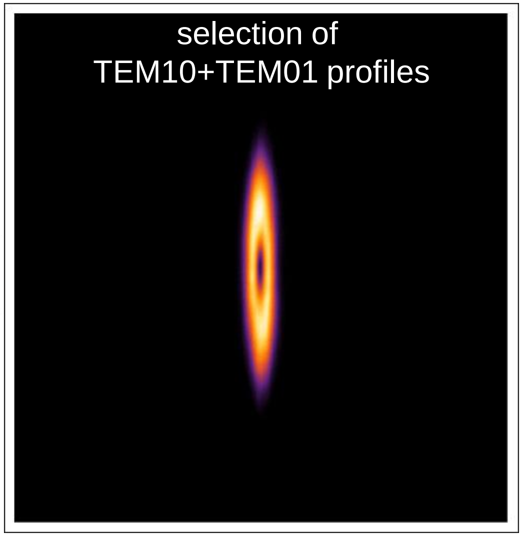
4D measurement

→ covers the whole phase space of the *non*-separable Wigner distribution

Outlook

4D Wigner measurement





- Nd:YAG laser with adjustable resonator
- modes: TEM00, TEM10+TEM01, TEM20 ⟶

- 50 z-positions, 15 rotation angles

4D Wigner measurement

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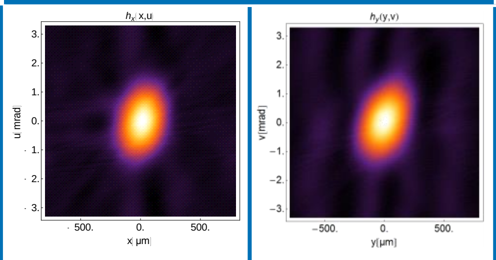
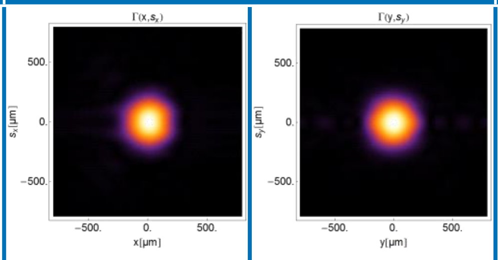
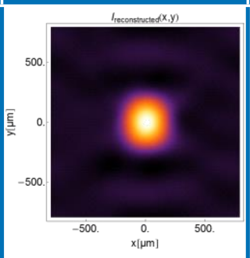
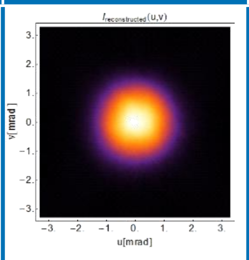
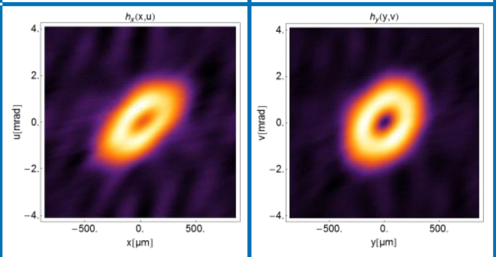
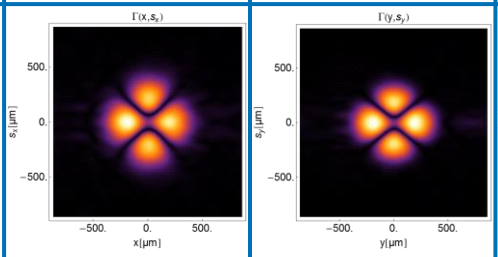
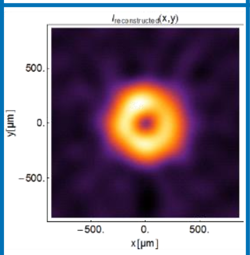
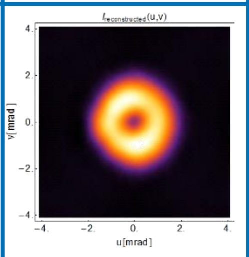
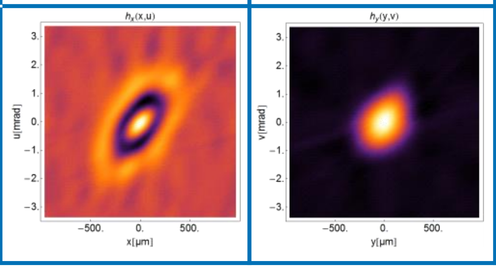
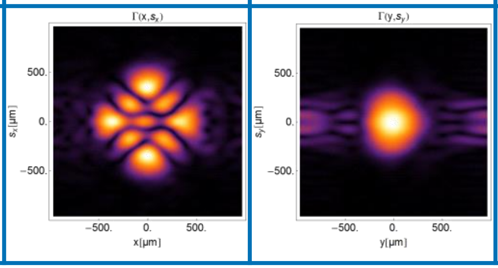
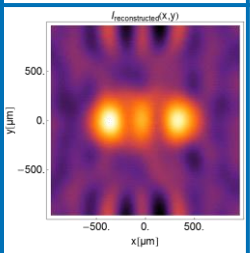
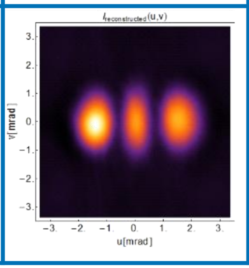
	Projected Wigner distribution	Mutual coherence function	Near field	Far field	Global degree of coherence
TEM00					***K* = 0.99 (1.00)**
TEM10 + TEM01					***K* = 0.46 (0.50)**
TEM20					***K* = 0.96 (1.00)**

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funding by

SFB 755

Nanoscale Photonic Imaging

Thank you!

[1] B. Flöter *et al*, *Beam parameters of FLASH beamline BL1 from Hartmann wavefront measurements*, Nucl. Instrum. Meth. A 635 (2011) 5108-5112
[2] B. Flöter *et al*, *EUV Hartmann sensor for wavefront measurements at the Free-electron LASer in Hamburg*, New J. Phys. 12 (2010) 083015
[3] A.E. Siegman, *ABCD-matrix elements for a curved diffraction grating,* J. Opt. Am. A 2 (1985) 1793
[4] M. Born and B. Wolf, *Principles of Optics* (1980) Cambridge University Press
[5] A. Singer *et al*, *Spatial and temporal coherence properties of single free-electron laser pulses* (2012) *arXiv:1206.1091*
[6] M. J. Bastiaans, *Wigner distribution function and its application to first-order optics* J. Opt. Soc. Am. 69 (1979) 1710-1716

Needed matrices :

general 4 x 4 Matrix of a gaussian beam:

$$G = \begin{pmatrix} \langle x^2 \rangle & 0 & \langle xu \rangle & 0 \\ 0 & \langle y^2 \rangle & 0 & \langle yv \rangle \\ \langle xu \rangle & 0 & \langle u^2 \rangle & 0 \\ 0 & \langle yv \rangle & 0 & \langle v^2 \rangle \end{pmatrix}; \; G_0 = \begin{pmatrix} {x_0}^2 & 0 & 0 & 0 \\ 0 & {x_0}^2 & 0 & 0 \\ 0 & 0 & {u_0}^2 & 0 \\ 0 & 0 & 0 & {u_0}^2 \end{pmatrix}$$

assumption for our HHG beam (focus position = plane wavefront = <xu> = <yv> = 0)

4 x 4 Matrix for propagation by z:

$$M_{\text{prop}}[z] = \begin{pmatrix} 1 & 0 & z & 0 \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4 x 4 Matrix for diffraction by a toroidal grating [1]:

$$M_{\text{grating}}[\alpha, \beta] = \begin{pmatrix} M & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2/R_t & 0 & 1/M & 0 \\ 0 & -2/R_s & 0 & 1 \end{pmatrix}$$

[1] A.E. Siegman, ABCD-matrix elements for a curved diffraction grating, 1985, J. Opt. Am. A

$$R_t = \frac{2\cos\alpha\cos\beta}{\cos\alpha + \cos\beta} R_x \qquad M = \frac{\cos\beta}{\cos\alpha}$$

$$R_s = \frac{2}{\cos\alpha + \cos\beta} R_y \qquad \beta[\alpha] = \sin^{-1}\left(\frac{N \cdot \lambda}{d} - \sin\alpha\right)$$

$x_0 =$ beam size, $u_0 =$ divergence, $\alpha =$ incidence angle, $\beta =$ diffraction angle,
$R_x =$ tangential radius, $R_y =$ sagittal radius, $N =$ diffraction order, $d =$ groove distance
$\lambda =$ wavelength

calculation:

$$G[\alpha] = M_{\text{prop}}[1319.9\text{mm}] \cdot M_{\text{grating}}[\alpha, \beta[\alpha]] \cdot M_{\text{prop}}[319.9\text{mm}] \cdot G_0$$

- $G[\alpha]$: resulting gaussian beam
- $M_{\text{prop}}[1319.9\text{mm}]$: propagation from grating to measurement position
- $M_{\text{grating}}[\alpha, \beta[\alpha]]$: grating diffracts beam into N = -1. order ($R_x = 1000\text{mm}$, $R_y = 104.09\text{mm}$, $d = 1\text{mm}/550$, $\lambda = 32\text{nm}$)
- $M_{\text{prop}}[319.9\text{mm}]$: propagation from source position to grating
- G_0: assuming HHG beam to be gaussian ($x_0 = 80\mu\text{m}$, $u_0 = 1.7$ mrad)

the curvature κ of the resulting wavefront is derived by:

$$\kappa_x = \frac{\langle xu \rangle}{\langle x^2 \rangle} \qquad \kappa_y = \frac{\langle yv \rangle}{\langle y^2 \rangle}$$

...and the radius of the wavefront by:

$$r_x = \frac{1}{\kappa_x} \qquad r_y = \frac{1}{\kappa_y}$$

waist difference:

$$\Delta z = \frac{\langle yv \rangle}{\langle v^2 \rangle} - \frac{\langle xu \rangle}{\langle u^2 \rangle}$$

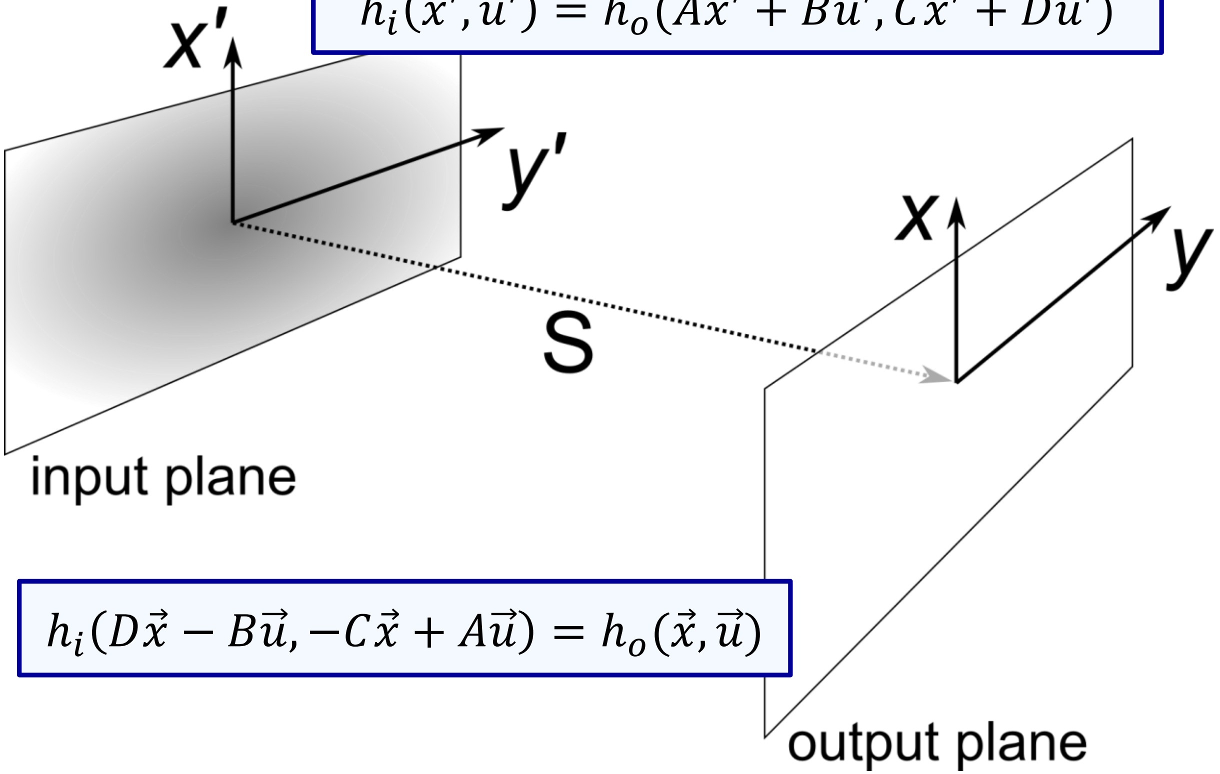
...yielding the radii difference:

$$\Delta r = \frac{\langle y^2 \rangle}{\langle yv \rangle} - \frac{\langle x^2 \rangle}{\langle xu \rangle}$$

Wigner distribution function

Beam propagation

$$h_i(\vec{x}', \vec{u}') = h_o(A\vec{x}' + B\vec{u}', C\vec{x}' + D\vec{u}')$$



$$h_i(D\vec{x} - B\vec{u}, -C\vec{x} + A\vec{u}) = h_o(\vec{x}, \vec{u})$$

Transformation

$$\begin{pmatrix} \vec{x} \\ \vec{u} \end{pmatrix} = S \cdot \begin{pmatrix} \vec{x}' \\ \vec{u}' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} \vec{x}' \\ \vec{u}' \end{pmatrix}$$

Back-transformation

$$\begin{pmatrix} \vec{x}' \\ \vec{u}' \end{pmatrix} = S^{-1} \cdot \begin{pmatrix} \vec{x} \\ \vec{u} \end{pmatrix} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \cdot \begin{pmatrix} \vec{x} \\ \vec{u} \end{pmatrix}$$

Free propagation

$$S = \begin{pmatrix} \mathbb{1} & z \cdot \mathbb{1} \\ 0 & \mathbb{1} \end{pmatrix}$$

Outlook

Detector saturation